
**Electronic Journal of Applied Statistical Analysis**
**EJASA, Electron. J. App. Stat. Anal.**
http://siba-ese.unisalento.it/index.php/ejasa/index
e-ISSN: 2070-5948
DOI: 10.1285/i20705948v13n2p293

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Published: 14 October 2020

*Electronic Journal of Applied Statistical Analysis*
Vol. 13, Issue 02, October 2020, 293-349
DOI: 10.1285/i20705948v13n2p293

# A new generalized log-logistic Erlang truncated exponential distribution with applications

Broderick Oluyede*[a], Hameed A. Jimoh[b], Divine Wanduku[c], and Boikanyo Makubate[d]

[a]*Department of Mathematical Sciences, Georgia Southern University, Statesboro, GA, 30460, USA.*
[b]*Department of Mathematical Sciences, Georgia Southern University, Statesboro, GA, 30460, USA.*
[c]*Department of Mathematical Sciences, Georgia Southern University, Statesboro, GA, 30460, USA.*
[d]*Department of Mathematics and Statistical Sciences, Botswana International University of Science and Technology, Palapye, BW.*

Published: 14 October 2020

We introduce a new distribution via the Marshall-Olkin generator called the Marshall-Olkin Log-logistic Erlang-Truncated Exponential (MOLLoGETE) distribution. Some structural properties of the distribution including series expansion of the density function, sub-models, hazard function, moments, conditional moments, mean deviations, distribution of order statistics, Rényi entropy and maximum likelihood estimates are presented. The new density function is an infinite linear combinations of Burr XII-Erlang-Truncated Exponential distributions. The new generalization is applied to real data sets to evaluate the model performance.

**keywords:** Marshall-Olkin,Generalized distribution, Erlang Truncated Exponential distribution, Maximum Likelihood Estimation.

*Corresponding author: boluyede@georgiasouthern.edu.


ISSN: 2070-5948
http://siba-ese.unisalento.it/index.php/ejasa/index

## 1 Introduction

Marshall and Olkin (1997) introduced a method of developing new distribution via addition of a new shape parameter. In the literature on statistical distribution, there are several ways of generating new families including the beta-G family (Lee et al., 2007), odd log-logistic-G family (Gleaton and Lynch, 2006), gamma-G family (Zografos and Balakrishnan, 2009), and Marshall-Olkin (Marshall and Olkin, 1997) generator among several other techniques.

Let $Y_1, Y_2, \cdots, Y_N$ be a sequence of independent and identically distributed random variable with cumulative distribution function (cdf) $F(x)$ and $N$ a random variable with probability mass function $P(N < n) = \delta(1 - \delta^{n-1})$, for $n = 1, 2, \cdots$, then the distribution of $X_N = \min(Y_1, Y_2, \cdots, Y_N)$ is given by equation (1), that is,

$$\begin{aligned} P(X_N \leq x) &= \sum_{n=1}^{\infty} P(X_N \leq x | N = n) P(N = n) \\ &= 1 - \frac{\delta \overline{F}(x)}{1 - \overline{\delta} \overline{F}(x)}, \end{aligned} \tag{1}$$

for $\delta > 0$, $\overline{\delta} = 1 - \delta$, where $\overline{F}(x) = 1 - F(x)$ is the survival or reliability function.

This work employs the Marshall-Olkin transformation to the log-Logistic Erlang truncated exponential distribution to obtain a new more flexible distribution for describing survival and reliability data. Marshall and Olkin applied the transformation and generalized the exponential and Weibull distribution. Subsequently, the Marshall-Olkin transformation was applied to Weibull distribution (Ghitany et al., 2005, Zhang and Xie, 2007). More recently, general results have been addressed by Barreto-Souza et al. (2013) and Cordeiro and Lemonte (2011). Santos-Nero et al. (2014) introduced a new class of models called the Marshall-Olkin extended Weibull family of distributions which defines several special models. Lepetu et al. (2016) developed the Marshall-Olkin Log-Logistic Extended Weibull distribution, that defines several new models and are applicable to several areas including survival and reliability analysis. However, these authors do not employ the Marshall-Olkin transformation in extending the Log-Logistic Erlang truncated exponential distribution. We note that the log-logistic Erlang truncated exponential distribution is also a new distribution by itself.

Erlang-truncated exponential (ETE) distribution is an extension of the well known exponential distribution and its cdf is given by

$$F(x) = 1 - e^{-\beta(1-e^{-\lambda})x}, \tag{2}$$

for $x \geq 0, \beta, \lambda > 0$. See El-Alosey (2007) for additional details. The application of ETE distribution is limited since it reduces to the exponential distribution as $\lambda \to \infty$. The ETE distribution is not suitable for modeling systems with non-monotone hazard rate function. We consider the generalization of ETE distribution via competing risk model (Oluyede et al., 2016). Consider a series system and let the lifetime of the components follow the log-logistic and ETE distributions with reliability functions $R_1(t) = (1+t^c)^{-1}$

and $R_2(t) = e^{-\beta(1-e^{-\lambda})t}$, respectively. Then the reliability of the system is

$$R(t) = (1 + t^c)^{-1} e^{-\beta(1-e^{-\lambda})t}. \tag{3}$$

The corresponding cdf is given by

$$F(t; c, \beta, \lambda) = 1 - (1 + t^c)^{-1} e^{-\beta(1-e^{-\lambda})t}, \tag{4}$$

for $c, \beta, \lambda > 0$ and $t \geq 0$. We refer to the distribution in equation (4) as the log-logistic Erlang truncated exponential (LLoGETE) distribution. We develop and study an extension of the LLoGETE distribution that is obtained via the Marshall-Olkin generator.

This paper is organized as follows. In Section 2, we present the new model, the Marshall-Olkin Log-logistic Erlang truncated exponential (MOLLoGETE) distribution and its sub-models. Section 2 also contain the statistical properties of MOLLoGETE distribution including the expansion of the density function and hazard function. Moments, conditional moments, mean deviation, Bonferroni and Lorenz curves, order statistics and Rényi entropy are presented in Section 3. The maximum likelihood estimates of the model parameters and asymptotic confidence intervals are discussed in Section 4. Section 5 is concerned with Monte Carlo simulations. Sections 6 and 7 contain applications and conclusions, respectively.

## 2 The Model

In this Section, we present the new Marshall-Olkin log-logistic Erlang truncated exponential (MOLLoGETE) distribution and its statistical properties, including expansion of density function, quantile function, sub-models and hazard function. The family of Marshall-Olkin (MO) (Marshall and Olkin, 1997) distributions has cdf given by

$$G(x; \delta) = 1 - \frac{\delta \overline{F}(x)}{1 - \overline{\delta}\overline{F}(x)} = \frac{F(x)}{1 - \overline{\delta}\overline{F}(x)}, \tag{5}$$

where $\delta$ is the tail (shape) parameter, and $\overline{\delta} = 1 - \delta$. The corresponding pdf of the MO distribution is given by

$$g(x; \delta) = \frac{\delta f(x)}{(1 - \overline{\delta}\overline{F}(x))^2}. \tag{6}$$

Applying MO distribution to the LLOGETE distribution, we obtain the cdf of Marshall-Olkin log-logistic Erlang-truncated exponential (MOLLoGETE) distribution given by

$$G_{MOLLoGETE}(x; \beta, \delta, c, \lambda) = \frac{1 - (1 + x^c)^{-1} e^{-\beta(1-e^{-\lambda})x}}{1 - \overline{\delta}(1 + x^c)^{-1} e^{-\beta(1-e^{-\lambda})x}}, \tag{7}$$

for $\beta, \delta, c, \lambda > 0$. The corresponding probability density function (pdf) is given by

$$g_{MOLLoGETE}(x; \beta, \delta, c, \lambda) = \frac{\delta e^{-\beta(1-e^{-\lambda})x}(1 + x^c)^{-2}(cx^{c-1} + (1 + x^c)\beta(1 - e^{-\lambda}))}{(1 - \overline{\delta}(1 + x^c)^{-1} e^{-\beta(1-e^{-\lambda})x})^2}, \tag{8}$$

for $\beta, \delta, c,$ and $\lambda > 0$. For simplicity, we let the pdf $g_{MOLLoGETE}(x; \beta, \delta, c, \lambda) = g(x)$ and cdf $G_{MOLLoGETE}(x; \beta, \delta, c, \lambda) = G(x)$.

Several plots of pdf of MOLLoGETE distribution for selected parameter values was given by Figure 1. The plot suggests that the MOLLoGETE pdf can be right skewed or decreasing for the selected values of the model parameters.

## 2.1 Series Expansion of Density Function

Applying the generalized binomial expansion (Gradshteyn and Ryzhik, 2000)

$$(1-z)^{-k} = \sum_{j=0}^{\infty} \frac{\Gamma(k+j)}{\Gamma(k)j!} z^j, \tag{9}$$

for $|z| < 1$, the pdf of MOLLoGETE distribution can be expressed as

$$\begin{aligned} g(x) &= \sum_{j=0}^{\infty} \frac{\Gamma(j+2)}{\Gamma(2)j!} \delta\bar{\delta}^j \left(e^{-\beta(1-e^{-\lambda})x}(1+x^c)^{-1}\right)^{j+1} \left(\beta(1-e^{-\lambda}) + cx^{c-1}(1+x^c)^{-1}\right) \\ &= \sum_{j=0}^{\infty} \delta\bar{\delta}^j g_*(x; c, j+1, \lambda, \beta(j+1)), \end{aligned} \tag{10}$$

where $g_*(x; c, j+1, \lambda, \beta(j+1))$ is the pdf of the Burr XII-Erlang-truncated exponential (Burr XII-ETE) distribution with parameters $c$, $\beta(j+1)$, $\lambda$ and $j+1 > 0$. The pdf of MOLLoGETE distribution can be written as an infinite linear combination of Burr XII-ETE distribution. Thus, the mathematical and statistical properties of the MOLLoGETE distribution follows from those of the Burr XII-ETE distribution.

## 2.2 Sub-models

Some new and known sub-models of MOLLoGETE distribution are presented in this Subsection.

1. When $\beta \to 0$, we obtain Marshall-Olkin Log-Logistic (MOLLoG) distribution with the cdf
$$G(x) = \frac{1-(1+x^c)^{-1}}{1-\bar{\delta}(1+x^c)^{-1}}.$$

2. When $\lambda \to \infty$, we obtain new Marshall-Olkin log-logistic exponential (MOLLoGE) distribution with the cdf
$$G(x) = \frac{1-(1+x^c)^{-1}e^{-\beta x}}{1-\bar{\delta}(1+x^c)^{-1}e^{-\beta x}}.$$

3. When $\delta = 1$, we obtain the new Log-Logistic Erlang truncated exponential (LLoGETE) distribution with the cdf
$$G(x) = 1-(1+x^c)^{-1}e^{-\beta(1-e^{-\lambda})x}.$$

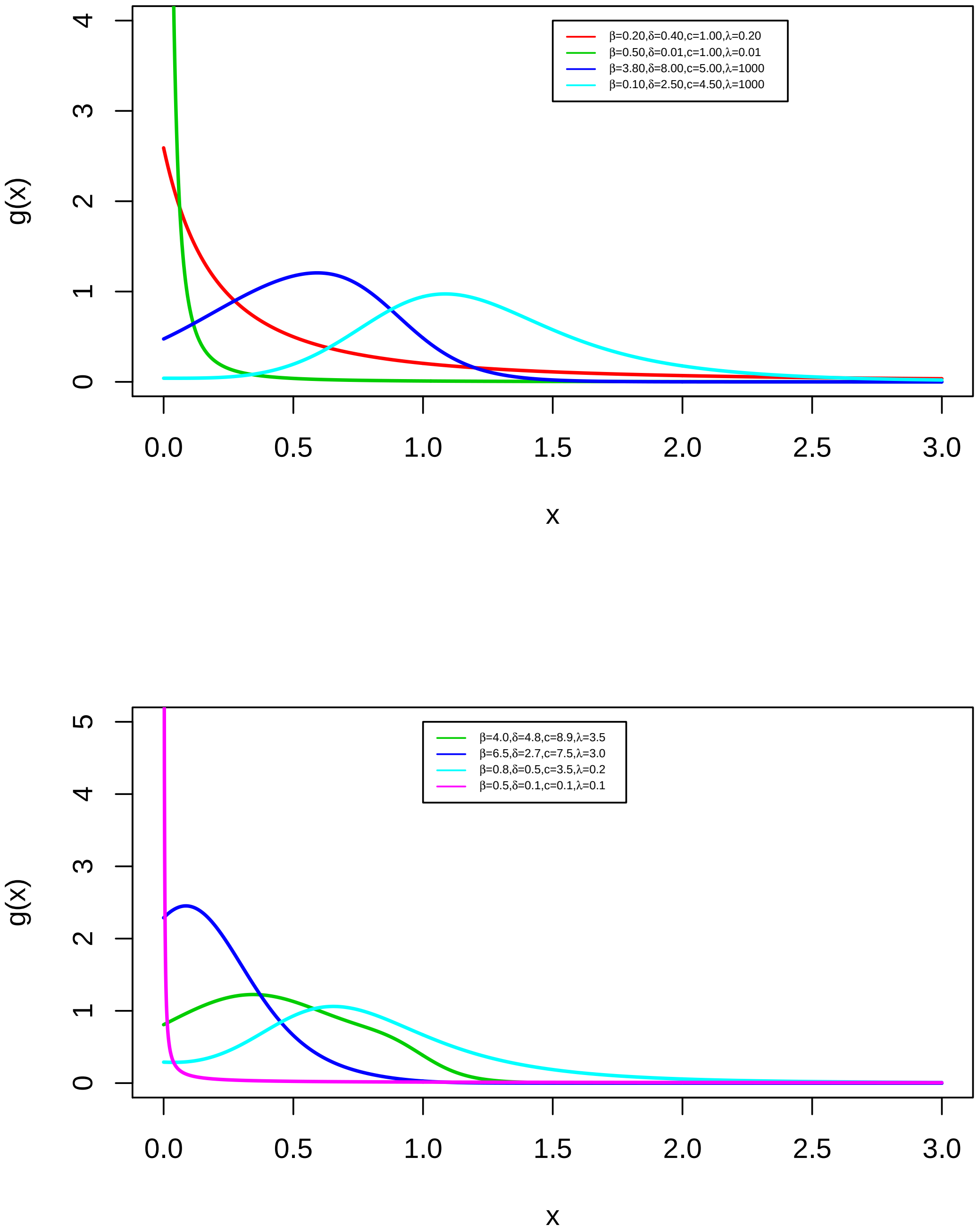



Figure 1: The graphs of pdf of MOLLoGETE distribution

4. When $\lambda \to 0$, we obtain the Marshall-Olkin log-logistic (MOLLoG) distribution with the cdf
$$G(x) = \frac{1-(1+x^c)^{-1}}{1-\overline{\delta}(1+x^c)^{-1}}.$$

5. When $c = 1$, we obtain a new distribution with the cdf
$$G(x) = \frac{1-(1+x)^{-1}e^{-\beta(1-e^{-\lambda})x}}{1-\overline{\delta}(1+x)^{-1}e^{-\beta(1-e^{-\lambda})x}}.$$

6. When $\beta = 1$, we obtain a new distribution with the cdf
$$G(x) = \frac{1-(1+x^c)^{-1}e^{-(1-e^{-\lambda})x}}{1-\overline{\delta}(1+x^c)^{-1}e^{-(1-e^{-\lambda})x}}.$$

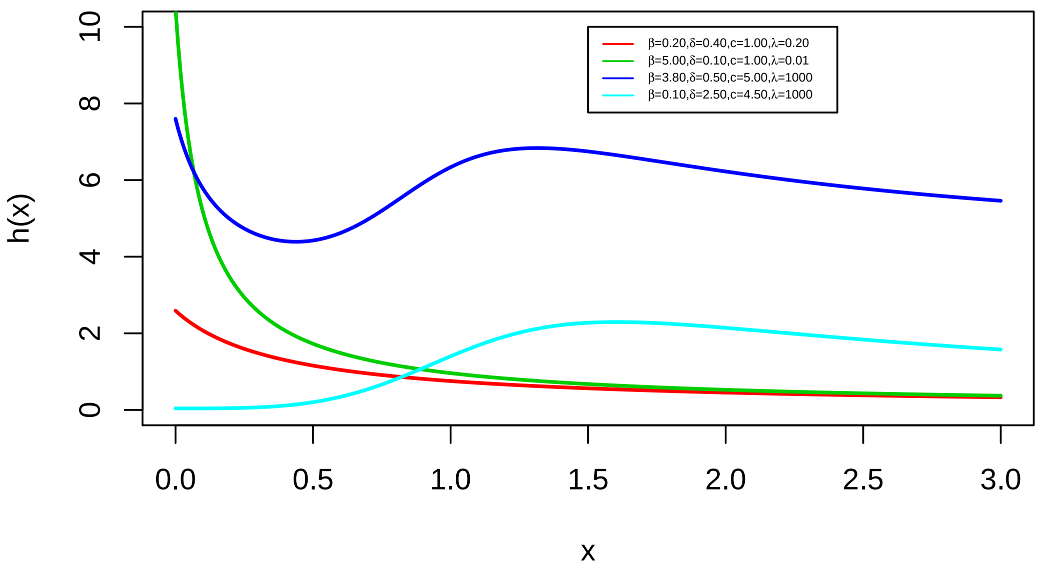
### 2.3 Hazard Function

The hazard function of MOLLoGETE distribution is given by
$$h_G(x) = \frac{e^{-\beta(1-e^{-\lambda})x}(1+x^c)^{-2}(cx^{c-1}+(1+x^c)\beta(1-e^{-\lambda}))}{\delta(1-(1+x^c)^{-1}e^{\beta(1-e^{-\lambda})x})(1-\overline{\delta}(1+x^c)^{-1}e^{\beta(1-e^{-\lambda})x})}. \quad (11)$$

Plots of the MOLLoGETE hazard function are given in Figure 2. These graphs exhibit uni-modal, decreasing, bathtub followed by upside down bathtub, upside down bathtub shape for the selected values of the model parameters. This flexibility makes the MOLLoGETE hazard function useful for non-monotonic empirical hazard behavior which are more likely to be encountered in practice or real life situations.

### 2.4 Quantile Function

The quantile function can be obtain by solving the equation
$$\frac{\delta(1+x^c)^{-1}e^{-\beta(1-e^{-\lambda})x}}{1-\overline{\delta}(1+x^c)^{-1}e^{-\beta(1-e^{-\lambda})x}} = 1-u \quad (12)$$
for $0 < u < 1$. That is,
$$(1-u)(1-\overline{\delta}(1+x^c)^{-1}e^{-\beta(1-e^{-\lambda})x}) - \delta(1+x^c)^{-1}e^{-\beta(1-e^{-\lambda})x} = 0,$$
and
$$(1-u) - \frac{(1-u)\overline{\delta}}{1+x^c}e^{-\beta(1-e^{-\lambda})x} - \frac{\delta}{1+x^c}e^{-\beta(1-e^{-\lambda})x} = 0.$$
It follows that
$$(1-u) - \frac{\delta+(1-u)\overline{\delta}}{1+x^c}e^{-\beta(1-e^{-\lambda})x} = 0,$$
and
$$\log(1-u) - \log\left(\frac{\delta+(1-u)\overline{\delta}}{1+x^c}e^{-\beta(1-e^{-\lambda})x}\right) = 0.$$



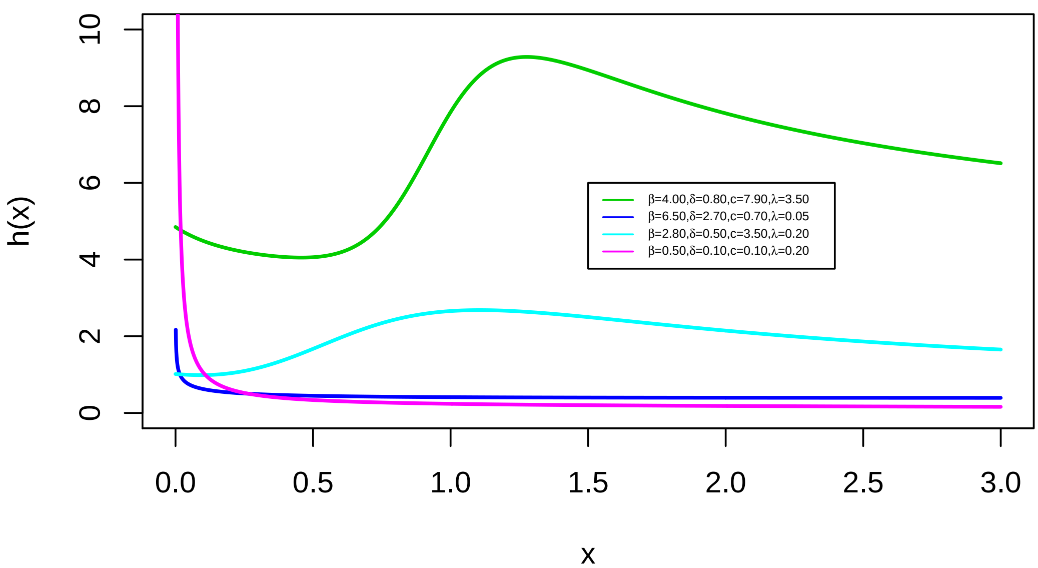




Figure 2: The graphs of hazard function of MOLLoGETE distribution

Consequently, the solution of the nonlinear equation

$$\log(1-u)-\Big(\log(\delta+(1-u)\overline{\delta})-\beta(1-e^{-\lambda})x-\log(1+x^c)\Big)=0 \tag{13}$$

yield the quantiles of MOLLoGETE distribution. Table 1 present quantiles for selected values of the parameters of the MOLLoGETE distribution.

Table 1: MOLLoGETE quantiles for selected parameter values

| u | (3,2,1,2) | (0.5,0.9,1,0.2) | (2,5,6,7) | (1.2,0.5,1,0.7) | (0.3,0.7,0.5,0.1) |
|---|---|---|---|---|---|
| 0.1 | 0.0563 | 0.0910 | 0.2211 | 0.0341 | 0.0060 |
| 0.2 | 0.1145 | 0.2027 | 0.4037 | 0.0751 | 0.0303 |
| 0.3 | 0.1761 | 0.3433 | 0.5582 | 0.1256 | 0.0881 |
| 0.4 | 0.2428 | 0.5256 | 0.6847 | 0.1893 | 0.2097 |
| 0.5 | 0.3173 | 0.7717 | 0.7887 | 0.2724 | 0.4595 |
| 0.6 | 0.4037 | 1.1226 | 0.8801 | 0.3860 | 0.9862 |
| 0.7 | 0.5099 | 1.6656 | 0.9689 | 0.5521 | 2.1750 |
| 0.8 | 0.6532 | 2.6258 | 1.0681 | 0.8238 | 5.1853 |
| 0.9 | 0.8897 | 4.8586 | 1.2084 | 1.3839 | 14.5656 |

# 3 Moments, Conditional Moments, Mean deviations, Bonferroni and Lorenz Curves, Order Statistics and Rényi Entropy

In this Section, moments, conditional moments, mean and median deviations, Bonferroni and Lorenz Curves, distribution of order Statistics and Rényi entropy for the MOLLoGETE distribution are presented. These measures (moments, mean and median deviations) can be readily obtained for the sub-models given in Section 2.

## 3.1 Moments

In this Subsection, we present the moments of the MOLLoGETE distribution. The $r^{th}$ non-negative integer moment of the MOLLoGETE distribution is given by

$$\begin{aligned} E[X^r] &= \int_0^\infty x^r g(x)dx \\ &= \sum_{j=0}^{\infty}\delta\overline{\delta}^j(j+1)\int_0^\infty x^r(1+x^c)^{-(j+1)}e^{-\beta(j+1)(1-e^{-\lambda})x} \\ &\times \Big(\beta(1-e^{-\lambda})+cx^{c-1}(1+x^c)^{-1}\Big)\,dx. \end{aligned} \tag{14}$$

Now, by applying power series expansion,

$$e^{-\beta(j+1)(1-e^{-\lambda})x} = \sum_{k=0}^{\infty} \frac{(-1)^k(\beta(j+1)(1-e^{-\lambda})x)^k}{k!}, \tag{15}$$

and let $t = (1+x^c)^{-1}$, then $x = \left(\frac{1-t}{t}\right)^{\frac{1}{c}}$ and $dx = -\frac{1}{c}\frac{1}{t^2}\left(\frac{1-t}{t}\right)^{1-1/c}$. We note that $t \to 1$ as $x \to 0$, and $t \to 0$ as $x \to \infty$. Thus,

$$E[X^r] = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^k(j+1)^{k+1}(1-e^{-\lambda})^k\beta^k}{k!}\delta\bar{\delta}^j \left[\frac{\beta(1-e^{-\lambda})}{c}\int_0^1 t^{-\frac{1}{c}(r+k+1)+j+1-1}\right.$$
$$\left.(1-t)^{\frac{1}{c}(r+k+1)-1}dt + \int_0^1 t^{-\frac{1}{c}(r+k+c)+j+2-1}(1-t)^{\frac{1}{c}(r+k+c)-1}dt\right]. \tag{16}$$

Thus, the $r^{th}$ moment of MOLLoGETE distribution reduces to

$$\begin{aligned} E[X^r] = &\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^k(j+1)^{k+1}(1-e^{-\lambda})^k\beta^k}{k!}\delta\bar{\delta}^j \\ &\times \left(\frac{\beta(1-e^{-\lambda})}{c}B\left(-\frac{1}{c}(r+k+1)+j+1, \frac{1}{c}(r+k+1)\right)\right. \\ &\left. + B\left(-\frac{1}{c}(r+k+c)+j+2, \frac{1}{c}(r+k+c)\right)\right), \end{aligned} \tag{17}$$

where $B(a,b) = \int_0^1 u^{a-1}(1-u)^{b-1}du$ is the beta function.

### 3.2 Conditional Moment

Conditional moments are very useful in survival and reliability analysis for lifetime models. They can also be used to obtain the mean deviations, Bonferroni and Lorenz curves. The $r^{th}$ non-negative integer conditional moment of the MOLLoGETE distribution is given by

$$E(X^r|X>t) = \frac{1}{\overline{G}(t)}\int_t^{\infty} x^r g(x)dx \tag{18}$$

$$\begin{aligned} &= \frac{1}{\overline{G}(t)}\sum_{j=0}^{\infty}\delta\bar{\delta}^j(j+1)\int_t^{\infty} x^r(1+x^c)^{-(j+1)}e^{-\beta(j+1)(1-e^{-\lambda})x} \\ &\times \left(\beta(1-e^{-\lambda}) + cx^{c-1}(1+x^c)^{-1}\right)dx. \end{aligned} \tag{19}$$

By applying power series expansion,

$$e^{-\beta(j+1)(1-e^{-\lambda})x} = \sum_{k=0}^{\infty} \frac{(-1)^k(\beta(j+1)(1-e^{-\lambda})x)^k}{k!}, \tag{20}$$

we have

$$E(X^r|X>t)=\frac{1}{\overline{G}(t)}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k(j+1)^{k+1}(1-e^{-\lambda})^k\beta^k}{k!}\delta\overline{\delta}^j\left(\beta(1-e^{-\lambda})\right.$$
$$\left.\times\int_t^{\infty}x^{r+k}(1+x^c)^{-(j+1)}dx+c\int_t^{\infty}x^{r+k+c-1}(1+x^c)^{-(j+1)}dx\right). \quad (21)$$

Let $t=(1+x^c)^{-1}$, $x=\left(\frac{1-y}{y}\right)^{\frac{1}{c}}$ and $dx=-\frac{1}{c}\frac{1}{y^2}\left(\frac{1-y}{y}\right)^{1-1/c}$. We note that $y\to 1(1+t^c)^{-1}$ as $x\to t$, and $y\to 0$ as $x\to\infty$. Thus,

$$E(X^r|X>t)=\frac{1}{\overline{G}(t)}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k(j+1)^{k+1}(1-e^{-\lambda})^k\beta^k}{k!}\delta\overline{\delta}^j\left[\frac{\beta(1-e^{-\lambda})}{c}\right.$$
$$\times\int_0^{(1+t^c)^{-1}}y^{-\frac{1}{c}(r+k+1)+j+1-1}(1-y)^{\frac{1}{c}(r+k+1)-1}dy$$
$$\left.+\int_0^{(1+t^c)^{-1}}y^{-\frac{1}{c}(r+k+c)+j+2-1}(1-y)^{\frac{1}{c}(r+k+c)-1}dy\right]. \quad (22)$$

Thus, the $r^{th}$ conditional moment of MOLLoGETE distribution is given by

$$E(X^r|X>t)=\frac{1}{\overline{G}(t)}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k(j+1)^{k+1}(1-e^{-\lambda})^k\beta^k}{k!}\delta\overline{\delta}^j$$
$$\times\left(\frac{\beta(1-e^{-\lambda})}{c}B_{(1+t^c)^{-1}}\left(-\frac{1}{c}(r+k+1)+j+1,\frac{1}{c}(r+k+1)\right)\right.$$
$$\left.+B_{(1+t^c)^{-1}}\left(-\frac{1}{c}(r+k+c)+j+2,\frac{1}{c}(r+k+c)\right)\right), \quad (23)$$

where $B_z(a,b)=\int_0^z u^{a-1}(1-u)^{b-1}$ is the incomplete beta function.

## 3.3 Mean Deviation

The mean deviation about the mean and the mean deviation about the median are defined by

$$\delta_1(x)=\int_0^{\infty}|x-\mu|g(x)dx \quad \text{and} \quad \delta_2(x)=\int_0^{\infty}|x-M|g(x)dx,$$

respectively, where $\mu=E[X]$ and $M=Median(X)$ denotes the median. Note that $\delta_1(x)$ and $\delta_2(x)$ can be obtained using the relationship

$$\delta_1(x)=2\mu G(\mu)-2\mu+2T(\mu) \quad \text{and} \quad \delta_2(x)=2T(M)-\mu, \quad (24)$$

$$\begin{aligned}
T(\mu) &= \int_{\mu}^{\infty} xg(x)dx \\
&= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^k (j+1)^{k+1}(1-e^{-\lambda})^k \beta^k}{k!} \delta\bar{\delta}^j \\
&\times \left( \frac{\beta(1-e^{-\lambda})}{c} B_{(1+\mu^c)^{-1}} \left( -\frac{1}{c}(k+2)+j+1, \frac{1}{c}(k+2) \right) \right. \\
&\left. + B_{(1+\mu^c)^{-1}} \left( -\frac{1}{c}(k+c+1)+j+2, \frac{1}{c}(k+c+1) \right) \right). \qquad (25)
\end{aligned}$$

## 3.4 Bonferroni and Lorenz Curves

Bonferroni and Lorenz curves of the MOLLoGETE distribution are given by

$$B(p) = \frac{1}{p\mu} \int_0^q xg(x)dx = \frac{1}{p\mu}[\mu - T(q)]$$

and

$$L(p) = \frac{1}{\mu} \int_0^q xg(x)dx = \frac{1}{\mu}[\mu - T(q)],$$

respectively, where

$$\begin{aligned}
T(q) &= \int_{q}^{\infty} xg(x)dx \\
&= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^t (j+1)^{k+1}(1-e^{-\lambda})^k \beta^k}{k!} \delta\bar{\delta}^j \\
&\times \left( \frac{\beta(1-e^{-\lambda})}{c} B_{(1+q^c)^{-1}} \left( \frac{1}{c}(k+2)+j+1, \frac{1}{c}(k+2) \right) \right. \\
&\left. + B_{(1+q^c)^{-1}} \left( -\frac{1}{c}(k+c+1)+j+2, \frac{1}{c}(k+c+1) \right) \right). \qquad (26)
\end{aligned}$$

## 3.5 Distribution of Order Statistics

Order statistics and entropy play important roles in probability and statistics, particularly in reliability, lifetime data analysis and information theory. In this Section, we present the distribution of the $i^{th}$ order statistic and Rényi entropy for the MOLLoGETE distribution. The pdf of the $i^{th}$ non-negative integer order statistics of a distribution with cdf $G(x)$ and pdf $g(x)$ is define as

$$\begin{aligned}
g_{i:n}(x) &= \frac{n!g(x)}{(i-1)!(n-i)!}[G(x)]^{i-1}[1-G(x)]^{n-i} \\
&= \frac{n!g(x)}{(i-1)!(n-i)!} \sum_{j=1}^{n-i} \binom{n-i}{j} (-1)^j (G(x))^{j+i-1}. \qquad (27)
\end{aligned}$$

The pdf of the $i^{th}$ order statistic for MOLLoGETE distribution is given by

$$\begin{aligned}
g_{i:n}(x) &= \frac{n!g(x)}{(i-1)!(n-i)!}\sum_{j=0}^{n-i}\binom{n-i}{j}(-1)^j\left(\frac{1-(1+x^c)^{-1}e^{-\beta(1-e^{-\lambda})x}}{1-\overline{\delta}(1+x^c)^{-1}e^{-\beta(1-e^{-\lambda})x}}\right)^{j+i-1}\\
&= \frac{n!g(x)}{(i-1)!(n-i)!}\sum_{j=0}^{n-1}\sum_{k=0}^{\infty}(-1)^{j+k}\binom{n-1}{j}\binom{j+i+k-2}{k}\overline{\delta}^k\\
&\times\left((1+x^c)^{-1}e^{-\beta(1-e^{-\lambda})x}\right)^k\sum_{m=0}^{j+i-1}\binom{j+i-1}{m}(-1)^m\left(1+x^c)^{-1}e^{-\beta(1-e^{-\lambda})x}\right)^m.
\end{aligned} \tag{28}$$

Substituting for the pdf $g(x)$, we have

$$\begin{aligned}
g_{i:n}(x) &= \frac{n!}{(i-1)!(n-i)!}\sum_{j=0}^{n-1}\sum_{k=0}^{\infty}\sum_{m=0}^{j+i-1}(-1)^{j+k+m}\binom{n-1}{j}\binom{j+i+k-2}{k}\binom{j+i-1}{m}\\
&\times\overline{\delta}^k\left(1+x^c)^{-1}e^{-\beta(1-e^{-\lambda})x}\right)^{k+m+1}(cx^{c-1}(1+x^c)^{-1}+\beta(1-e^{-\lambda}))\\
&\times(1-\overline{\delta}(1+x^c)^{-1}e^{-\beta(1-e^{-\lambda})x})^{-2}\\
&= \frac{n!}{(i-1)!(n-i)!}\sum_{j=0}^{n-1}\sum_{k,n=0}^{\infty}\sum_{m=0}^{j+i-1}(-1)^{j+k+m}\binom{n-1}{j}\binom{j+i+k-2}{k}\\
&\times\binom{j+i-1}{m}\binom{n+1}{n}\overline{\delta}^{k+n}\left((1+x^c)^{-1}e^{-\beta(1-e^{-\lambda})x}\right)^{k+m+n+1}\\
&\times(cx^{c-1}(1+x^c)^{-1}+\beta(1-e^{-\lambda}))\frac{k+m+n+1}{k+m+n+1}
\end{aligned} \tag{29}$$

which is an infinite sum of the pdf of Burr XII-ETE distribution with parameter $k+m+n+1>0$, $\beta(k+m+n+1)>0, c>0$ and $\lambda\geq 0$.Thus, the pdf of the $i^{th}$ order statistics can be expressed as a linear combination of the pdf of Burr XII-ETE distribution.

## 3.6 Rényi Entropy

Rényi entropy (Rényi, 1960) is an extension of Shannon entropy. Rényi entropy of MOLLoGETE distribution (for $v>0$, and $v\neq 1$, and the logarithm with base 2) is given by

$$\begin{aligned}
I_R(v) &= \frac{1}{1-v}\log\left(\int_0^\infty (g(x))^v\,dx\right)\\
&= \frac{1}{1-v}\log\left(\int_0^\infty e^{-\beta v(1-e^{-\lambda})x}(1+x^c)^{-v}\left(cx^{c-1}(1+x^c)^{-1}+\beta(1-e^{-\lambda})\right)^v\right.\\
&\times\left.\left(1-\overline{\delta}e^{-\beta(1-e^{-\lambda})x}(1+x^c)^{-1}\right)^{-2v}dx\right).
\end{aligned} \tag{30}$$

We note that

$$e^{-\beta v(1-e^{-\lambda})x} = \sum_{k=0}^{\infty} \frac{(-1)^k \beta^k v^k (1-e^{-\lambda})^k x^k}{k!}, \tag{31}$$

$$\left(cx^{c-1}(1+x^c)^{-1} + \beta(1-e^{-\lambda})\right)^v = \sum_{j=0}^{v} \binom{v}{j} c^{v-j} x^{(c-1)(v-j)} (1+x^c)^{-(v-j)} \beta^j (1-e^{-\lambda})^j, \tag{32}$$

and

$$\begin{aligned}\left(1-\overline{\delta} e^{-\beta(1-e^{-\lambda})x}(1+x^c)^{-1}\right)^{-2v} &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \binom{2v+m-1}{m} \overline{\delta}^m (1+x^c)^{-m} \\ &\times \frac{(-1)^n \beta^n m^n (1-e^{-\lambda})^n x^n}{n!}.\end{aligned} \tag{33}$$

Now substituting equations (31), (32) and (33) in equation (30), and let $t = (1+x^c)^{-1}$, then $x = \left(\frac{1-t}{t}\right)^{\frac{1}{c}}$ and $dx = -\frac{1}{c}\frac{1}{t^2}\left(\frac{1-t}{t}\right)^{1-1/c} dt$. We note that $t \to 1$ as $x \to 0$, and $t \to 0$ as $x \to \infty$. Thus, we have

$$\begin{aligned} I_R(v) = \frac{1}{1-v} \log &\Bigg( \sum_{k,m,n=0}^{\infty} \sum_{j=0}^{v} \binom{v}{j} \binom{2v+m-1}{m} \overline{\delta}^m \beta^{k+j+n} v^k c^{v-j-1} m^n \\ &\times \frac{(1-e^{-\lambda})^{k+j+n}}{k!n!} \int_0^1 t^{-\frac{1}{c}(k+(c-1)(v-j)+n-1)+j-2v-m-2-1} \\ &\times (1-t)^{\frac{1}{c}(k+(c-1)(v-j)+n-1)-1} dt \Bigg). \end{aligned} \tag{34}$$

Consequently, Rényi entropy of MOLLoGETE distribution is

$$\begin{aligned} I_R(v) = \frac{1}{1-v} \log &\Bigg( \sum_{k,m,n=0}^{\infty} \sum_{j=0}^{v} \binom{v}{j} \binom{2v+m-1}{m} (-1)^{k+n} \overline{\delta}^m \beta^{k+j+n} v^k c^{v-j-1} m^n \\ &\times \frac{(1-e^{-\lambda})^{k+j+n}}{k!n!} B\Bigg(\frac{1}{c}(k+(c-1)(v-j)+n-1)+j-2v-m-2, \\ &\frac{1}{c}(k+(c-1)(v-j)+n-1)\Bigg)\Bigg), \end{aligned} \tag{35}$$

for $v > 0, v \neq 1$. Rényi entropy tends to Shannon entropy as $v \to 1$.

# 4 Maximum Likelihood Estimates

In this Section, the maximum likelihood estimates and asymptotic confidence intervals are discussed. Let $X \sim MOLLoGETE(\beta, \delta, c, \lambda)$ and $\triangle = (\beta, \delta, c, \lambda)$ the parameter vector. The log-likelihood for a single observation $x$ of $X$ is given by

$$\ell = \ell(\triangle) = \log(\delta) + \log(cx^{c-1}(1+x^c)^{-1} + \beta(1-e^{-\lambda})) - \log(1+x^c) - \beta(1-e^{-\lambda})x - 2\log(1-\overline{\delta}(1+x^c)^{-1}e^{-\beta(1-e^{-\lambda})x}). \quad (36)$$

The partial derivative of the log-likelihood function with respect to the elements of the parameter vector $\triangle$ are given by

$$\frac{\partial\ell}{\partial\beta} = \frac{1-e^{-\lambda}}{cx^{c-1}(1+x^c)^{-1}+\beta(1-e^{-\lambda})} - (1-e^{-\lambda}) + \frac{2\overline{\delta}(1+x^c)^{-1}(1-e^{-\lambda})xe^{-\beta(1-e^{-\lambda})x}}{1-\overline{\delta}(1+x^c)^{-1}e^{-\beta(1-e^{-\lambda})x}},$$

$$\frac{\partial\ell}{\partial\delta} = \frac{1}{\delta} - \frac{2(1+x^c)^{-1}e^{-\beta(1-e^{-\lambda})x}}{1-\overline{\delta}(1+x^c)^{-1}e^{-\beta(1-e^{-\lambda})x}}$$

$$\frac{\partial\ell}{\partial c} = \frac{c(c-1)x^{c-2}(1+x^c)^{-1} - c^2x^{2(c-1)}(1+x^c)^{-2}}{cx^{c-1}(1+x^c)^{-1}+\beta(1-e^{-\lambda})} - \frac{x^c\ln(x)}{1+x^c} - \frac{2\overline{\delta}(1+x^c)^{-2}e^{-\beta(1-e^{-\lambda})x}x^c\ln(x)}{1-\overline{\delta}(1+x^c)^{-2}e^{-\beta(1-e^{-\lambda})x}},$$

and

$$\frac{\partial\ell}{\partial\lambda} = \frac{2\overline{\delta}\beta x^2(1+x^c)^{-1}e^{-\lambda}(1-e^{-\lambda})e^{-\beta(1-e^{-\lambda})x}}{1-\overline{\delta}(1+x^c)^{-1}e^{-\beta(1-e^{-\lambda})x}} + \beta e^{-\lambda} + \frac{\beta e^{-\lambda}}{cx^{c-1}(1+x^c)^{-1}+\beta(1-e^{-\lambda})},$$

respectively. These equations are solved numerically to obtain $\hat{\beta}$, $\hat{\delta}$ , $\hat{c}$ and $\hat{\lambda}$, the maximum likelihood estimates of the parameters $\beta$, $\delta$, $c$, and $\lambda$ respectively.

## 4.1 Asymptotic Confidence Interval

In this Section, we present the asymptotic confidence intervals of the parameters of the MOLLoGETE distribution. The expectations of the Fisher Information Matrix (FIM) can be obtained numerically. Let $\hat{\triangle} = (\hat{\beta}, \hat{\delta}, \hat{c}, \hat{\lambda})$ be the maximum likelihood estimate of $\triangle = (\beta, \delta, c, \lambda)$. Under the usual regularity conditions and that the parameters are in the interior of the parameter space, but not on the boundary, we have: $\sqrt{n}(\hat{\triangle} - \triangle) \sim N_4(\underline{0}, I^{-1}(\triangle))$, where $I(\triangle)$ is the expected Fisher Information Matrix. The asymptotic behavior is still valid if $I(\triangle)$ is replaced by the observed information matrix evaluates at $\hat{\triangle}$, that is $J(\hat{\triangle})$. The multivariate normal distribution $N_4(\underline{0}, I^{-1}(\triangle))$, where the mean vector $\underline{0} = (0,0,0,0)^T$, can be used to construct confidence interval and confidence regions for individual model parameters and for the survival and hazard rate functions. The approximate $100(1-\alpha)\%$ two-sided confidence intervals for $\beta, \lambda, c, \delta$ are given by: $\hat{\beta} \pm Z_{\frac{\alpha}{2}}\sqrt{I_{\beta\beta}^{-1}(\hat{\triangle})}$, $\hat{\delta} \pm Z_{\frac{\alpha}{2}}\sqrt{I_{\delta\delta}^{-1}(\hat{\triangle})}$ , $\hat{c} \pm Z_{\frac{\alpha}{2}}\sqrt{I_{cc}^{-1}(\hat{\triangle})}$ and $\hat{\lambda} \pm Z_{\frac{\alpha}{2}}\sqrt{I_{\lambda\lambda}^{-1}(\hat{\triangle})}$, respectively, where $I_{\beta\beta}^{-1}(\hat{\triangle})$, $I_{\delta\delta}^{-1}(\hat{\triangle})$, $I_{cc}^{-1}(\hat{\triangle})$ and $I_{\lambda\lambda}^{-1}(\hat{\triangle})$ are the diagonal elements of $I_n^{-1}(\hat{\triangle}) = (nI(\hat{\triangle}))^{-1}$, and $Z_{\frac{\alpha}{2}}$ is the upper $\alpha^{th}$ percentile of a standard normal distribution.

We can use the likelihood ratio (LR) test to compare the fit of the MOLLoGETE distribution with its sub-models for a given data set. For example, to test $\delta = 1$, the LR statistic is

$$w = 2\Big( \ln\big(L(\hat{\beta}, \hat{\delta}, \hat{c}, \hat{\lambda})\big) - \ln\big(L(\tilde{\beta}, 1, \tilde{c}, \tilde{\lambda})\big)\Big),$$

where $\hat{\beta}, \hat{\delta}, \hat{c}$, and $\hat{\lambda}$ are the unrestricted estimates, and $\tilde{\beta}, \tilde{c}$, and $\tilde{\lambda}$ are the restricted estimates. The LR test rejects the null hypothesis if $w > \chi^2_d$, where $\chi^2_d$ denote the upper $100\%d$ point of the $\chi^2$ distribution with 1 degree of freedom.

## 5 Monte-Carlo Simulation

We investigate the accuracy and performance of the maximum likelihood estimates of the MOLLoGETE model parameters by conducting various simulations for different sample sizes and different parameter values. Equation (13) is used to generate random data from MOLLoGETE distribution. The simulation study is repeated $N = 2000$ times each with sample size $n = 30, 50, 100, 200, 400, 800$ with true values $I : \beta = 1.0, \delta = 1.0, c = 1.0, \lambda = 0.1$, and $II : \beta = 1.0, \delta = 1.0, c = 2.0, \lambda = 9.0$. The following quantities were computed in the simulation study.

1. Average bias of the MLE $\hat{\theta}$ of the parameter $\theta = \beta, c, \delta, \lambda$ :

$$AB(\hat{\theta}) = \frac{1}{N}\sum_{k=1}^{N}(\hat{\theta} - \theta).$$

2. Root mean squared error (RMSE) of the MLE $\hat{\theta}$ of the parameter $\theta = \beta, c, \delta, \lambda$ :

$$RMSE(\hat{\theta}) = \sqrt{\frac{1}{N}\sum_{k=1}^{N}(\hat{\theta} - \theta)^2}.$$

From Table 2, we observed that as sample size increases, the biases decreases for the parameter values. Also, the RMSE decreases as sample size increases.

## 6 Applications

In this Section, we present examples to illustrate the applicability and flexibility of the MOLLoGETE distribution and its sub-models for data modeling. The maximum likelihood estimates (MLEs) of MOLLoGETE model parameters $\triangle = (\beta, \delta, c, \lambda)$ are computed by maximizing the objective function via bbmle and mle2 in R.

These functions were applied and executed for a wide range of initial values. This process often results or lead to more than one maximum, however, in these case, we take MLEs corresponding to the largest value of the maxima. In a few cases, a new initial value was tried in order to obtain a maximum. The estimated value of the model parameters (standard error in parenthesis), $-2$ log-likelihood statistics, Akaike Information Criterion, $AIC = 2p - 2\ln(L)$, Bayesian Information Criterion, $BIC = p\ln(n) - 2\ln(L)$, and Consistent Akaike Information Criterion, $AICC = AIC + 2\frac{p(p+1)}{n-p-1}$, where $L = L(\hat{\triangle})$ is the value of the likelihood function evaluated at the parameter estimates, $n$ is the number of observations, and $p$ is the number of estimated parameter presented in Table 3 and 4. The Cramer-von Mises and Anderson-Darling goodness-of-fit statistics $W^*$ and $A^*$ described by Chen and Balakrishnan (1995) are also presented in the tables. These statistics can be used to verify which distribution fits better to the data. In general, the smaller the values of $W^*$ and $A^*$, the better the fit. The **AdequacyModel** package in R (R Development Core Team, 2011) was used to evaluate the fitted distributions.

The MOLLoGETE distribution was also compared to the non-nested gamma Dagum (GD) distribution (Oluyede et al., 2014) for the breaking strength of carbon fibre data (Nichols and Padgett, 2006). Also, we compared the fit of MOLLoGETE distribution with the fits of GD distribution as well as the fit of the Marshall-Olkin generalized Erlang-truncated exponential (MOGETE) distribution (Okorie et al., 2017) for bank customer waiting times data (Ghitany et al., 2008). The pdf of the GD distribution is given by

$$g_{GD}(x; \lambda, \beta, \delta, \alpha) = \frac{\lambda\delta\beta^{\alpha}}{\Gamma(\alpha)} x^{-\delta-1}(1 + \lambda x^{-\delta})^{-\beta-1}\left(\log\left(1 + \lambda x^{-\delta}\right)\right)^{\alpha-1}$$

for $\lambda, \beta, \delta, \alpha > 0$ and $x > 0$, and the pdf of the MOGETE distribution is given by

$$g_{MOGETE}(x; \alpha, \beta, \lambda) = \alpha\beta(1 - e^{-\lambda})e^{-\beta(1-e^{-\lambda})x}\left(1 - (1-\alpha)e^{-\beta(1-e^{-\lambda})x}\right)^{-2}$$

for $\alpha, \beta, \alpha > 0$ and $x > 0$.

### 6.1 Breaking Strength of Carbon Fibre Data

The data set below represent breaking strength of carbon fibre (Gba) reported in (Nichols and Padgett, 2006). The data set consists of 100 observations.

3.70 2.74 2.73 2.50 3.60 3.11 3.27 2.87 1.47 3.11 4.42 2.41 3.19 3.22 1.69 3.28
3.09 1.87 3.15 4.90 3.75 2.43 2.95 2.97 3.39 2.96 2.53 2.67 2.93 3.22 3.39 2.81
4.20 3.33 2.55 3.31 3.31 2.85 2.56 3.56 3.15 2.35 2.55 2.59 2.38 2.81 2.77 2.17
2.83 1.92 1.41 3.68 2.97 1.36 0.98 2.76 4.91 3.68 1.84 1.59 3.19 1.57 0.81 5.56
1.73 1.59 2.00 1.22 1.12 1.71 2.17 1.17 5.08 2.48 1.18 3.51 2.17 1.69 1.25 4.38
1.84 0.39 3.68 2.48 0.85 1.61 2.79 4.70 2.03 1.80 1.57 1.08 2.03 1.61 2.12 1.89
2.88 2.82 2.05 3.65

The 95% confidence intervals for the model parameters are given by $\beta \in (1.6003 \pm 1.96 * 7.1761), \delta \in (114.4006 \pm 1.96 * 63.6599), c \in (1.4149 \pm 1.96 * 1.2092)$ and $\lambda \in (1.4149 \pm 1.96 * 13.9366)$.

From Table 3, the likelihood ratio (LR) test statistics for testing the hypotheses $H_0$ : MOLLoGETE(1,$\delta$,$c$,$\lambda$) against $H_a$ : MOLLoGETE and $H_0$ : MOLLoGETE($\beta$,$\delta$,1,$\lambda$) against $H_a$ : MOLLoGETE ($w = 15.704$, p-value= $7.4067 \times 10^{-5}$, and $w = 26.6284$, p-value= $2.4659 \times 10^{-7}$) shows significance difference between the fits of MOLLoGETE and MOLLoGETE(1,$\delta$,$c$,$\lambda$) distributions as well as the fits between MOLLoGETE and MOLLoGETE($\beta$,$\delta$,1,$\lambda$) distributions. There is also significance difference between the fits of MOLLoGETE and MOLLoGETE(0,$\delta$,1,$\lambda$) distributions and that of fits between MOLLoGETE and LLoG distributions. Also, MOLLoGETE distribution is clearly a better fit than the non-nested GD distributions based on the goodness-of-fit statistics $A^*$ and $W^*$.

Plots of fitted densities and the histogram of the data are given in Figure 3. Probability plots (Chambers et al., 1983) are also present in Figure 4. For probability plot, we plotted $G(x_{(j)}; \hat{\beta}, \hat{\delta}, \hat{c}, \hat{\lambda})$ against $\frac{j-0.375}{n+0.25}, j = 1, 2, ..., n$, where $x_{(j)}$ are the ordered values of the observed data. The measures of closeness are given by the sum of squares

$$SS = \sum_{j=1}^{n} \left[ G(x_{(j)}; \hat{\beta}, \hat{\delta}, \hat{c}, \hat{\lambda}) - \left( \frac{j - 0.375}{n + 0.25} \right) \right]^2 .$$

The value of SS from the probability plot clearly shows that the MOLLoGETE distribution is the preferred fit for the data.

## 6.2 Bank Customers Waiting Times Data

The data set below represents the waiting times in minutes of 100 bank customers in a queue before service (Ghitany et al., 2008). The data are:
0.8 0.8 1.3 1.5 1.8 1.9 1.9 2.1 2.6 2.7 2.9 3.1 3.2 3.3 3.5 3.6 4.0 4.1 4.2 4.2 4.3 4.3 4.4 4.4 4.6 4.7 4.7 4.8 4.9 4.9 5.0 5.3 5.5 5.7 5.7 6.1 6.2 6.2 6.2 6.3 6.7 6.9 7.1 7.1 7.1 7.1 7.4 7.6 7.7 8.0 8.2 8.6 8.6 8.6 8.8 8.8 8.9 8.9 9.5 9.6 9.7 9.8 10.7 10.9 11.0 11.0 11.1 11.2 11.2 11.5 11.9 12.4 12.5 12.9 13.0 13.1 13.3 13.6 13.7 13.9 14.1 15.4 15.4 17.3 17.3 18.1 18.2 18.4 18.9 19.0 19.9 20.6 21.3 21.4 21.9 23.0 27.0 31.6 33.1 38.5.

We note that from Table 4, the LR test statistics for testing the hypotheses $H_0$ : MOLLoGETE(1,$\delta$,$c$,$\lambda$) against $H_a$ : MOLLoGETE and $H_0$ : MOLLoGETE($\beta$,$\delta$,1,$\lambda$) against $H_a$ : MOLLoGETE ($w = 57.5946$, p-value= $3.2211 \times 10^{-14}$, and $w = 67.8452$, p-value= $1.7685 \times 10^{-16}$) shows significance difference between the fits of MOLLoGETE and MOLLoGETE(1,$\delta$,$c$,$\lambda$) distributions as well as the fits between MOLLoGETE and MOLLoGETE($\beta$,$\delta$,1,$\lambda$) distributions. There is also significance difference between the fits of MOLLoGETE and MOLLoGETE(0,$\delta$,1,$\lambda$) distributions and that of fits between MOLLoGETE and LLoG distributions. The value of the goodness-of-fit statistics $A^*$ and $W^*$ shows that the MOLLoGETE distribution is the better fit compare to its sub-models. Clearly, MOLLoGETE distribution is a better fit than the non-nested GD and MOGETE distributions based on the goodness-of-fit statistics $A^*$ and $W^*$. Plots of

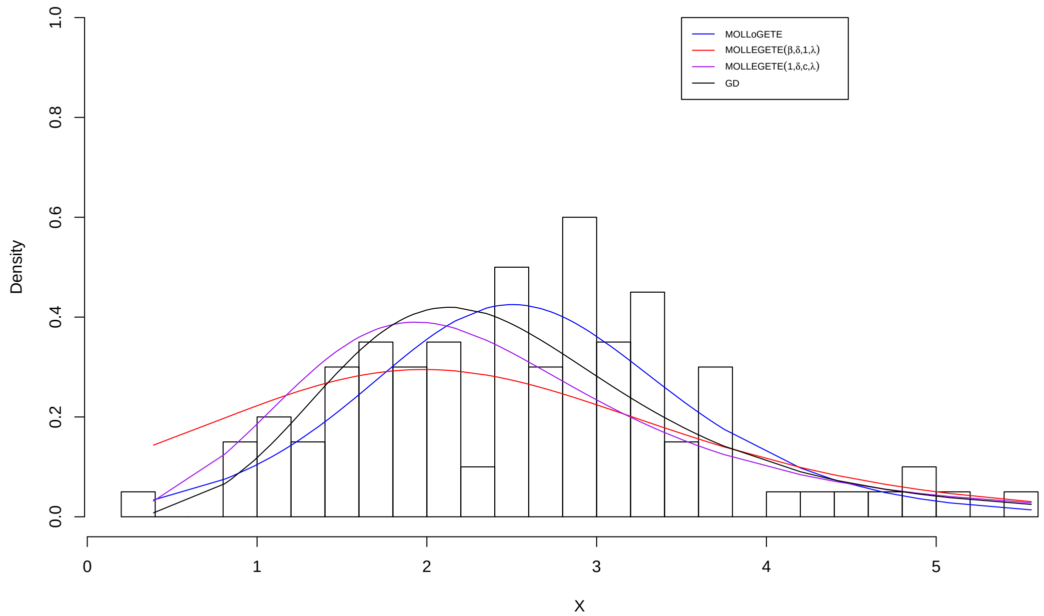



Figure 3: Fitted densities of MOLLoGETE distribution, sub-models and GD distribution

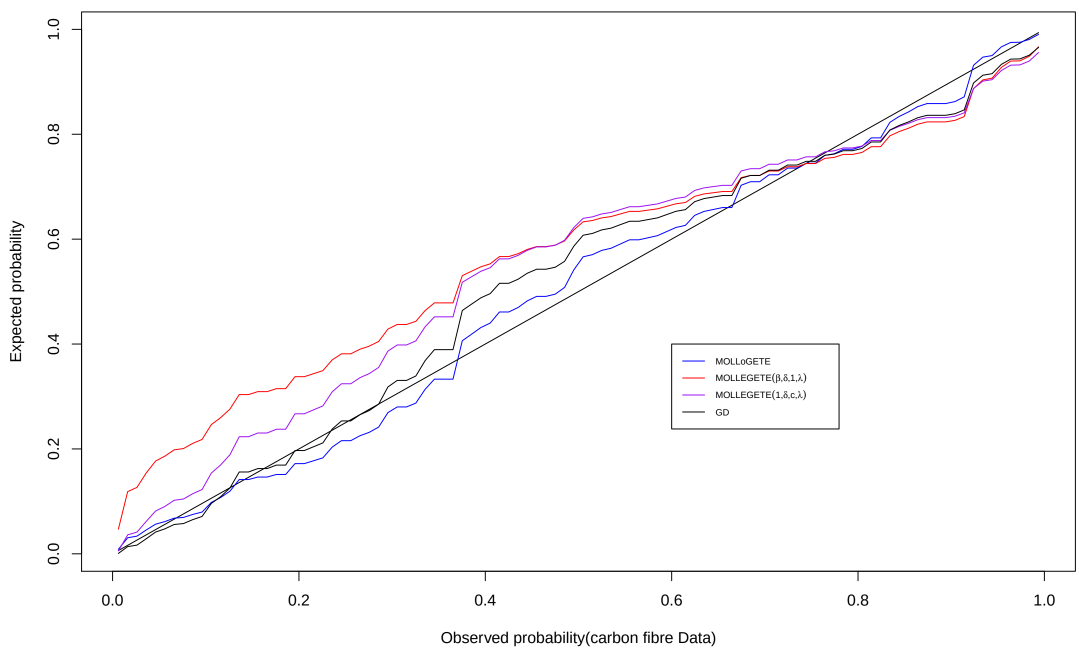



Figure 4: Probability plots of carbon fibre data

fitted densities and the histogram of the data are given in Figure 5. Probability plots (Chambers et al., 1983) are also present in figure 6 and point to the MOLLoGETE distribution as the better fit for the bank customer waiting times data.

## 7 Concluding Remarks

In this paper, we have proposed, developed and presented results on a new generalized distribution called MOLLoGETE distribution. Properties of the distribution including series expansion of the pdf, hazard function, quantile function, moments, conditional moments, inequality measures such as Lorenz and Bonferroni curves are derived. Rényi entropy, distribution of order statistics and maximum likelihood estimates of the model parameter are presented. A Monte Carlo simulation study was performed to examine the performance of the MOLLoGETE distribution. Applications of the model to real data are also presented.

## Acknowledgement

The authors are particularly grateful to the anonymous referees and Editor for their useful comments and suggestions.

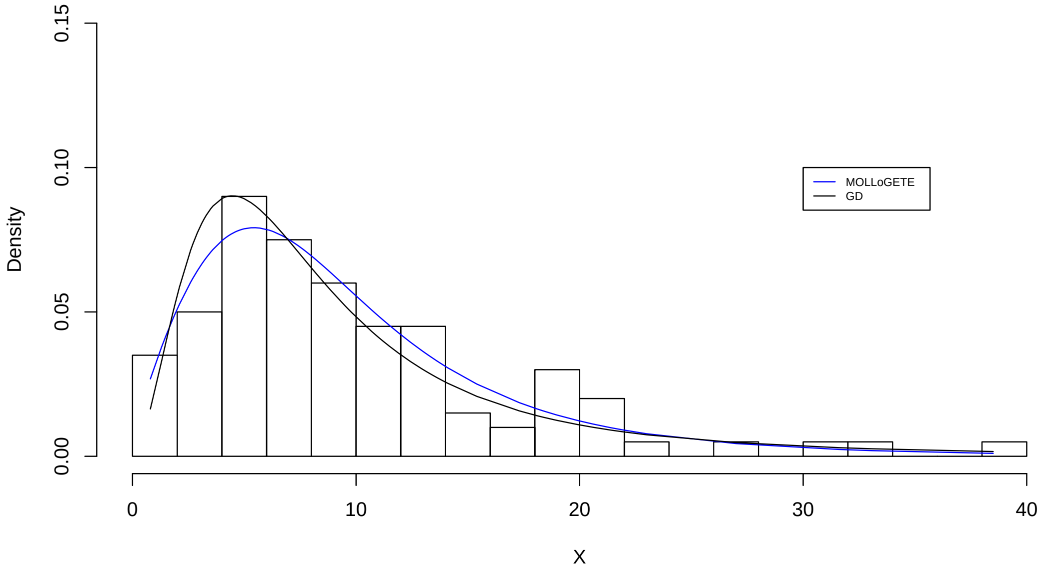



Figure 5: Fitted densities of MOLLoGETE distribution, sub-models and GD distribution

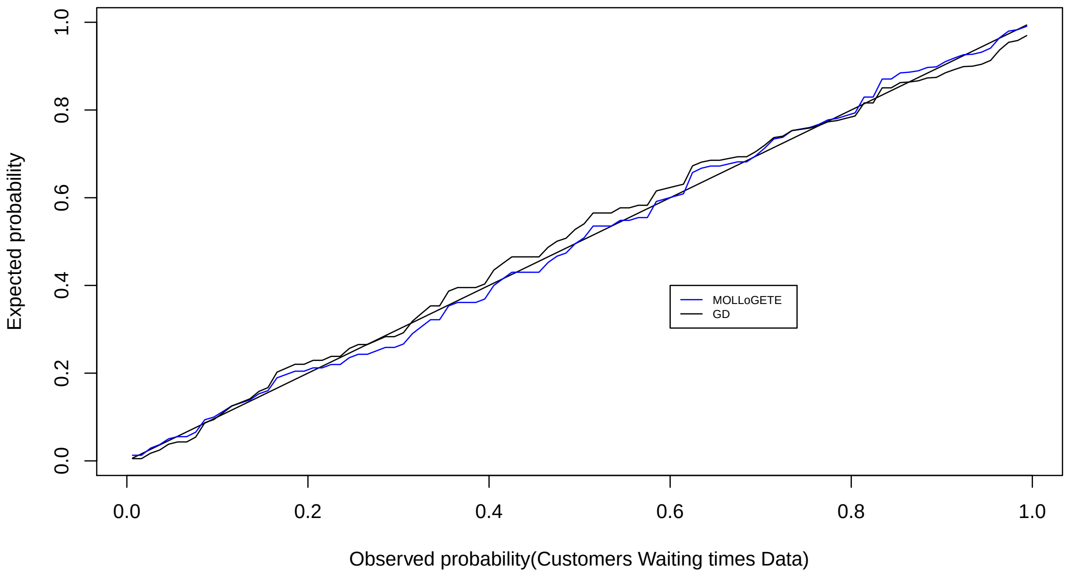



Figure 6: Probability plots of bank customers waiting times data

# Appendix 1

### Second partial derivatives of the log-likelihood functions

$$\frac{\partial^2 \ell}{\partial \beta^2} = -\frac{(1-e^{-\lambda})^2}{(cx^{c-1}(1+x^c)^{-1}+\beta(1-e^{-\lambda}))^2} - \frac{2\overline{\delta}(1+x^c)^{-1}(1-e^{-\lambda})^2 x^2 e^{-\beta(1-e^{-\lambda})x}}{(1-\overline{\delta}(1+x^c)^{-1}e^{-\beta(1-e^{-\lambda})x})^2},$$

$$\frac{\partial^2 \ell}{\partial \delta^2} = -\frac{1}{\delta^2} - \frac{2(1+x^c)^{-2}e^{-2\beta(1-e^{-\lambda})x}}{(1-\overline{\delta}(1+x^c)^{-1}e^{-\beta(1-e^{-\lambda})x})^2},$$

$$\begin{aligned}\frac{\partial^2 \ell}{\partial c^2} = &\Bigg\{\Big(cx^{c-1}+\beta(1-e^{-\lambda})(1+x^c)\Big)\Big((2c-1)x^{c-2}+(c^2-c)x^{c-2}\ln(x) \\ &- 2cx^{2(c-1)}(1+x^c)^{-1} - 2c^2x^{2(c-1)}\ln(x)(1+x^c)^{-1} + c^2x^{2(c-1)}x^c\ln(x)(1+x^c)^{-2}\Big) \\ &- \Big((c^2-c)x^{c-2} - c^2x^{2(c-1)}(1+x^c)^{-1}\Big)\Big(x^{c-1}+cx^{c-1}\ln(x)+\beta(1-e^{-\lambda})x^c\ln(x)\Big)\Bigg\} \\ &\div \Big(cx^{c-1}+\beta(1-e^{-\lambda})(1+x^c)\Big)^2 - \frac{x^c\ln^2(x)}{(1+x^c)^2} + \left(\frac{2\overline{\delta}e^{-\beta(1-e^{-\lambda})x}\ln(x)}{(1+x^c)^2(x^c+1)^2-\overline{\delta}e^{-\beta(1-e^{-\lambda})x})^2}\right) \\ &\times \Big(-\overline{\delta}e^{-\beta(1-e^{-\lambda})x}x^c\ln(x) + x^c\ln(x) - \overline{\delta}e^{-\beta(1-e^{-\lambda})x}x^{2c}\ln(x) - x^{3c}\ln(x) + x^{2c}\ln(x) \\ &- x^{4c}\ln(x)\Big),\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 \ell}{\partial \lambda^2} = &\,2\overline{\delta}\beta x^2(1+x^c)^{-1}e^{-\lambda}e^{-\beta(1-e^{-\lambda})x}\Bigg\{\Big(1-\overline{\delta}(1+x^c)^{-1}e^{-\beta(1-e^{-\lambda})x}\Big) \\ &\times \Big(2e^{-\lambda}-1-\beta xe^{-\lambda}(1-e^{-\lambda})\Big) - (1-e^{-\lambda})\Big(\overline{\delta}(1+x^c)^{-1}\beta xe^{-\lambda}e^{-\beta(1-e^{-\lambda})x}\Big)\Bigg\} \\ &\div \Big(1-\overline{\delta}(1+x^c)^{-1}e^{-\beta(1-e^{-\lambda})x}\Big)^2 - \beta e^{-\lambda} + \frac{\beta(1+x^c)\Big(ce^{-\lambda}x^{c-1}-\beta e^{-\lambda}(1+x^c)\Big)}{\Big(cx^{c-1}+\beta(1-e^{-\lambda})(1+x^c)\Big)^2},\end{aligned}$$

$$\frac{\partial^2 \ell}{\partial \beta \partial \delta} = -\frac{2(1+x^c)^{-1}x(1-e^{-\lambda})e^{-\beta(1-e^{-\lambda})x}}{\Big(1-\overline{\delta}(1+x^c)^{-1}e^{-\beta(1-e^{-\lambda})x}\Big)^2},$$

$$\frac{\partial^2\ell}{\partial\beta\partial c} = -\frac{\left(1-e^{-\lambda}\right)\left(x^{c-1}(1+x^c)^{-1} + cx^{c-1}\ln(x)(1+x^c)^{-2}\right.}{\left(cx^{c-1}(1+x^c)^{-1} + \beta(1-e^{-\lambda})\right)^2} - \frac{2\overline{\delta}(1+x^c)^{-2}(1-e^{-\lambda})x^{c+1}\ln(x)e^{-\beta(1-e^{-\lambda})x}}{\left(1-\overline{\delta}(1+x^c)^{-1}e^{-\beta(1-e^{-\lambda})x}\right)^2},$$

$$\frac{\partial^2\ell}{\partial\beta\partial\lambda} = \frac{cx^{c-1}(1+x^c)^{-1}e^{-\lambda}}{\left(cx^{c-1}(1+x^c)^{-1} + \beta(1-e^{-\lambda})\right)^2} - e^{-\lambda} + \frac{2\overline{\delta}(1+x^c)^{-1}xe^{-\beta(1-e^{-\lambda})x}}{\left(1-\overline{\delta}(1+x^c)^{-1}e^{-\beta(1-e^{-\lambda})x}\right)^2} \times \left(\overline{\delta}(1+x^c)^{-1}e^{-\lambda}e^{-\beta(1-e^{-\lambda})x}(1-\beta xe^{-\lambda}) + e^{-\lambda}(1-\beta x)\right),$$

$$\frac{\partial^2\ell}{\partial\delta\partial c} = \frac{2(1+x^c)^{-2}x^c\ln(x)e^{-\beta(1-e^{-\lambda})x}}{\left(1-\overline{\delta}(1+x^c)^{-1}e^{-\beta(1-e^{-\lambda})x}\right)^2},$$

$$\frac{\partial^2\ell}{\partial\delta\partial\lambda} = \frac{2(1+x^c)^{-1}\beta xe^{-\lambda}e^{-\beta(1-e^{-\lambda})x}}{\left(1-\overline{\delta}(1+x^c)^{-1}e^{-\beta(1-e^{-\lambda})x}\right)^2},$$

and

$$\frac{\partial^2\ell}{\partial c\partial\lambda} = -\frac{\beta e^{-\lambda}\left(c(c-1)x^{c-2}(1+x^c)^{-1} - c^2x^{2(c-1)}(1+x^c)^{-2}\right)}{\left(cx^{c-1}(1+x^c)^{-1} + \beta(1-e^{-\lambda})\right)^2} + \frac{2\overline{\delta}(1+x^c)^{-2}x^{c+1}\ln(x)\beta e^{-\lambda}e^{-\beta(1-e^{-\lambda})x}}{\left(1-\overline{\delta}(1+x^c)^{-2}e^{-\beta(1-e^{-\lambda})x}\right)^2}.$$

# Appendix 2

### Monte Carlo Simulations

```
rm(list=ls())
library(stats4)
library(bbmle)
library(stats)
library(numDeriv)

#define MOLLOGETE pdf
MOLLOGETE_pdf <- function(beta, delta, c, lambda, x){
  (exp(-beta*(1-exp(-lambda))*x)*(1+x^c)^(-2)*delta*(c*x^(c-1)+(1+x^c)*
      beta*(1-exp(-lambda))))/(1-(1-delta)*exp(-beta*(1-exp(-lambda))*x)
      *(1+x^c)^(-1))^2

}
#define MOLLOGETE Quantiles
MOLLOGETE_quantile <- function(beta, delta, c, lambda, u){
  f <- function(x){
    log((1-u)/(delta+(1-u)*(1-delta)))+beta*(1-exp(-lambda))*x+log(1+x^c)
  }
  rc <- uniroot(f, lower = 0, upper = 100, tol = 1e-9)
  result <- rc$root
}

##maximum likelihood estimation
MOLLOGETE_LL<-function(par){-sum(log((exp(-par[1]*(1-exp(-par[4]))*x)
    *(1+x^par[3])^(-2)*par[2]*(par[3]*x^(par[3]-1)+(1+x^par[3])*par
    [1]*(1-exp(-par[4]))))/(1-(1-par[2])*exp(-par[1]*(1-exp(-par[4]))*x)
    *(1+x^par[3])^(-1))^2))}


mle.result<-nlminb(c(beta, delta, c, lambda),MOLLOGETE_LL,lower=0,upper=
    Inf)

#Monte-Carlo simulation study

beta=1
delta=1
c=1
lambda=0.1

n1=c(1500)
```

```
for(j in 1:length(n1)){
  n=n1[j]
  N=800
  mle_beta<-c(rep(0,N))
  mle_delta<-c(rep(0,N))
  mle_c<-c(rep(0,N))
  mle_lambda<-c(rep(0,N))
  LC_beta<-c(rep(0,N))
  UC_beta<-c(rep(0,N))
  LC_delta<-c(rep(0,N))
  UC_delta<-c(rep(0,N))
  LC_c<-c(rep(0,N))
  UC_c<-c(rep(0,N))
  LC_lambda<-c(rep(0,N))
  UC_lambda<-c(rep(0,N))

  count_beta=0
  count_delta=0
  count_c=0
  count_lambda=0


  temp=1
  HH1<-matrix(c(rep(2,16)),nrow=4,ncol=4)
  HH2<-matrix(c(rep(2,16)),nrow=4,ncol=4)
  for(i in 1:N)
  {
    print(i)
    flush.console()
    repeat{
      x<-c(rep(0,n))

      #Generate a random variable from uniform distribution
      u<-0
      u<-runif(n,min=0,max=1)

      for (m in 1:n){
        x[m]<-MOLLOGETE_quantile(beta, delta, c, lambda,u[m])
      }

      #maximum likelihood estimation
      mle.result<-nlminb(c(beta, delta, c, lambda),
                         MOLLOGETE_LL,lower=0,upper=Inf)
```

```
  summary(mle.result)


  temp=mle.result$convergence
  if(temp==0){
    temp_beta<-mle.result$par[1]
    temp_delta<-mle.result$par[2]
    temp_c<-mle.result$par[3]
    temp_lambda<-mle.result$par[4]
    HH1<-hessian(MOLLOGETE_LL, c(temp_beta,temp_delta,temp_c,
        temp_lambda))
    if(sum(is.nan(HH1))==0 & (diag(HH1)[1]>0) & (diag(HH1)[2]>0) &(
        diag(HH1)[3]>0) &(diag(HH1)[4]>0) ) {
      HH2<-solve(HH1)
      #print(det(HH1))
    }
    else{
      temp=1}
  }
  if((temp==0)& (diag(HH2)[1]>0) &(diag(HH2)[2]>0) &(diag(HH2)[3]>0)
      &(diag(HH2)[4]>0) & (sum(is.nan(HH2))==0)){
    break
  }
  else{
    temp=1}
}
temp=1
mle_beta[i]<-mle.result$par[1]
mle_delta[i]<-mle.result$par[2]
mle_c[i]<-mle.result$par[3]
mle_lambda[i]<-mle.result$par[4]


HH<-hessian(MOLLOGETE_LL,c(mle_beta[i],mle_delta[i],mle_c[i],
    mle_lambda[i]))
H<-solve(HH)
LC_beta[i]<-mle_beta[i]-qnorm(0.975)*sqrt(diag(H)[1])
UC_beta[i]<-mle_beta[i]+qnorm(0.975)*sqrt(diag(H)[1])
if((LC_beta[i]<=beta) & (beta<=UC_beta[i])){
  count_beta=count_beta+1
}
LC_delta[i]<-mle_delta[i]-qnorm(0.975)*sqrt(diag(H)[2])
UC_delta[i]<-mle_delta[i]+qnorm(0.975)*sqrt(diag(H)[2])
if((LC_delta[i]<=delta) & (delta<=UC_delta[i])){
```

```
  count_delta=count_delta+1
}
LC_c[i]<-mle_c[i]-qnorm(0.975)*sqrt(diag(H)[3])
UC_c[i]<-mle_c[i]+qnorm(0.975)*sqrt(diag(H)[3])
if((LC_c[i]<=c) & (c<=UC_c[i])){
  count_c=count_c+1
}

LC_lambda[i]<-mle_lambda[i]-qnorm(0.975)*sqrt(diag(H)[4])
UC_lambda[i]<-mle_lambda[i]+qnorm(0.975)*sqrt(diag(H)[4])
if((LC_lambda[i]<=lambda) & (lambda<=UC_lambda[i])){
  count_lambda=count_lambda+1
}
#calculate Average Bias
ABias_beta<-sum(mle_beta-beta)/N
ABias_delta<-sum(mle_delta-delta)/N
ABias_c<-sum(mle_c-c)/N
ABias_lambda<-sum(mle_lambda-lambda)/N


print(cbind(ABias_beta,ABias_delta,ABias_c,ABias_lambda))

#Calculate RMSE

RMSE_beta<-sqrt(sum((beta-mle_beta)^2)/N)
RMSE_delta<-sqrt(sum((delta-mle_delta)^2)/N)
RMSE_c<-sqrt(sum((c-mle_c)^2)/N)
RMSE_lambda<-sqrt(sum((lambda-mle_lambda)^2)/N)

print(cbind(RMSE_beta,RMSE_delta,RMSE_c,RMSE_lambda))

#converge probability
CP_beta<-count_beta/N
CP_delta<-count_delta/N
CP_c<-count_c/N
CP_lambda<-count_lambda/N

print(cbind(CP_beta,CP_delta,CP_c,CP_lambda))

#average Width
AW_beta<-sum(abs(UC_beta-LC_beta))/N
AW_delta<-sum(abs(UC_delta-LC_delta))/N
AW_c<-sum(abs(UC_c-LC_c))/N
AW_lambda<-sum(abs(UC_lambda-LC_lambda))/N
```

```
    print(cbind(AW_beta,AW_delta,AW_c,AW_lambda))

  }
}
```

## Application 1

Carbon Fibre Data

```
require(gsl)
require(stats)

###Carbon Fibre Data########
data=c(3.70, 2.74, 2.73, 2.50, 3.60, 3.11, 3.27,

      2.87, 1.47, 3.11, 4.42, 2.41, 3.19, 3.22, 1.69,3.28, 3.09, 1.87,
         3.15, 4.90, 3.75,

      2.43, 2.95, 2.97, 3.39, 2.96, 2.53, 2.67, 2.93, 3.22, 3.39, 2.81,
         4.20, 3.33, 2.55,

      3.31, 3.31, 2.85, 2.56, 3.56, 3.15, 2.35, 2.55, 2.59, 2.38, 2.81,
         2.77, 2.17, 2.83,

      1.92, 1.41, 3.68, 2.97, 1.36, 0.98, 2.76, 4.91, 3.68, 1.84, 1.59,
         3.19, 1.57, 0.81,

      5.56, 1.73, 1.59, 2.00, 1.22, 1.12, 1.71, 2.17, 1.17, 5.08, 2.48,
         1.18, 3.51, 2.17,

      1.69, 1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.70,
         2.03, 1.80, 1.57,

      1.08, 2.03, 1.61, 2.12, 1.89, 2.88, 2.82, 2.05, 3.65)
data<-sort(data)
n=length(data)

####### Full MOLLEGETE Model ########
###### Parameter estimates ######
beta1=1.6003
delta1=114.4066
c1=1.4990
lambda1=1.4149
###### PDF #########
MOLLOGETE_pdf <- function(data,beta,delta,c,lambda,n) {
  f<-c(rep(0,n))
  for (i in 1:n)
  {
    x<-data[i]
    f[i]<-(exp(-beta*(1-exp(-lambda))*x)*(1+x^c)^(-2)*delta*(c*x^(c-1)
```

```
        +(1+x^c)*beta*(1-exp(-lambda))))/(1-(1-delta)*exp(-beta*(1-exp(-
        lambda))*x)*(1+x^c)^(-1))^2
  }
  return(f)
}
###### FIT ###
f1<-MOLLOGETE_pdf(data,beta1,delta1,c1,lambda1,n)

######### CDF #########
MOLLOGETE_cdf <- function(data,beta,delta,c,lambda,n){
  F<-c(rep(0,n))
  for (i in 1:n)
  {
    x<-data[i]
    F[i]<-(1-(1+x^c)^(-1)*exp(-beta*(1-exp(-lambda))*x))/(1-(1-delta)*(1+
        x^c)^(-1)*exp(-beta*(1-exp(-lambda))*x))
  }
  return(F)
}
########
F1<-MOLLOGETE_cdf(data,beta1,delta1,c1,lambda1,n)
F1


############# Sub model with c=1 ################
#####Parameter estimates#######
beta2 = 0.8532
delta2=18.0307
lambda2=2.5780

####### PDF #########
MOLLOGETE1_pdf <- function(data,beta,delta,lambda,n) {
  c=1
  f<-c(rep(0,n))
  for (i in 1:n)
  {
    x<-data[i]
    f[i]<-(exp(-beta*(1-exp(-lambda))*x)*(1+x^c)^(-2)*delta*(c*x^(c-1)
        +(1+x^c)*beta*(1-exp(-lambda))))/(1-(1-delta)*exp(-beta*(1-exp(-
        lambda))*x)*(1+x^c)^(-1))^2
  }
  return(f)
}
####### FITS #########
```

```
f2<-MOLLOGETE1_pdf(data,beta2,delta2,lambda2,n)

######### CDF #########
MOLLOGETE1_cdf <- function(data,beta,delta,lambda,n){
  c=1
  F<-c(rep(0,n))
  for (i in 1:n)
  {
    x<-data[i]
    F[i]<-(1-(1+x^c)^(-1)*exp(-beta*(1-exp(-lambda))*x))/(1-(1-delta)*(1+
        x^c)^(-1)*exp(-beta*(1-exp(-lambda))*x))
  }
  return(F)
}
########
F2<-MOLLOGETE1_cdf(data,beta2,delta2,lambda2,n)
F2


################# Sub Model with beta=1 ###############
###### Parameter estimates #####
delta3=17.3095
c3=3.1216
lambda3= 0.1080
####### PDF #########
MOLLOGETE2_pdf <- function(data,delta,c,lambda,n) {
  beta=1
  f<-c(rep(0,n))
  for (i in 1:n)
  {
    x<-data[i]
    f[i]<-(exp(-beta*(1-exp(-lambda))*x)*(1+x^c)^(-2)*delta*(c*x^(c-1)
        +(1+x^c)*beta*(1-exp(-lambda))))/(1-(1-delta)*exp(-beta*(1-exp(-
        lambda))*x)*(1+x^c)^(-1))^2
  }
  return(f)
}
####### FITS #########
f3<-MOLLOGETE2_pdf(data,delta3,c3,lambda3,n)

######### CDF #########
MOLLOGETE2_cdf <- function(data,delta,c,lambda,n){
  beta=1
  F<-c(rep(0,n))
```

```
  for (i in 1:n)
  {
    x<-data[i]
    F[i]<-(1-(1+x^c)^(-1)*exp(-beta*(1-exp(-lambda))*x))/(1-(1-delta)*(1+
        x^c)^(-1)*exp(-beta*(1-exp(-lambda))*x))
  }
  return(F)
}
########
F3<-MOLLOGETE2_cdf(data,delta3,c3,lambda3,n)
F3


#############Sub-models with beta->0 and c=1 #########
###### Parameter estimates #####
delta5=2.4192
lambda5=0.3452
####### PDF #########
MOLLOGETE4_pdf <- function(data,delta,lambda,n) {
  beta=0.00001
  c=1
  f<-c(rep(0,n))
  for (i in 1:n)
  {
    x<-data[i]
    f[i]<-(exp(-beta*(1-exp(-lambda))*x)*(1+x^c)^(-2)*delta*(c*x^(c-1)
        +(1+x^c)*beta*(1-exp(-lambda))))/(1-(1-delta)*exp(-beta*(1-exp(-
        lambda))*x)*(1+x^c)^(-1))^2
  }
  return(f)
}
####### FITS #########
f5<-MOLLOGETE4_pdf(data,delta5,lambda5,n)

######### CDF #########
MOLLOGETE2_cdf <- function(data,delta,c,lambda,n){
  beta=1
  F<-c(rep(0,n))
  for (i in 1:n)
  {
    x<-data[i]
    F[i]<-(1-(1+x^c)^(-1)*exp(-beta*(1-exp(-lambda))*x))/(1-(1-delta)*(1+
        x^c)^(-1)*exp(-beta*(1-exp(-lambda))*x))
  }
```

```
  return(F)
}
########
F3<-MOLLOGETE2_cdf(data,delta3,c3,lambda3,n)
F3

########## Non-nested Gamma Dagum model################
###### Parameter estimates #####
alpha4=0.2171
beta4=5.3891
delta4=3.4171
lambda4=20.0138
####### PDF #########
GD_pdf <- function(data,alpha,beta,delta,lambda,n) {
   f<-c(rep(0,n))
  for (i in 1:n)
  {
    x<-data[i]
    f[i]<-(1/gamma(alpha))*beta*delta*lambda*x^(-delta-1)*(1+lambda*x^(-
        delta))^(-beta-1)*(-log(1-(1+lambda*x^(-delta))^(-beta)))^(alpha
        -1)
  }
  return(f)
}
####### FITS #########
f4<-GD_pdf(data,alpha4,beta4,delta4,lambda4,n)

######### CDF #########
GD_cdf <- function(data,alpha,beta,delta,lambda,n){
  F<-c(rep(0,n))
  for (i in 1:n)
  {
    x<-data[i]
    F[i]<-pgamma(-log(1-(1+lambda*x^(-delta))^(-beta)),alpha)
  }
  return(F)
}
########
F4<-GD_cdf(data,alpha4,beta4,delta4,lambda4,n)
F4


######### Fitting the densities of the models #################
op <- par(mfrow=c(1,1))
```

```
X <- data
hist(X, prob=TRUE,ylim=c(0,1.0),main=' ', breaks = 20) # prob=TRUE for
   probabilities not counts # add a density estimate with defaults
lines(data,f1,col='blue',type='l')
lines(data,f2,col='red')
lines(data,f3,col='purple',type='l')
lines(data,f4,col='black',type='l')

legend(3.5,1.,pt.cex = 1,cex=0.6, # places a legend at the appropriate
   place
      c("MOLLoGETE",expression(MOLLEGETE(paste(beta,",",delta,",",1,",",
         lambda))),expression(MOLLEGETE(paste(1,",",delta,",",c,",",
         lambda))),"GD"), # puts text in the legend
      lty=c(1,1,1,1), # gives the legend appropriate symbols (lines)
      lwd=c(1,1,1,1),col=c("blue","red","purple","black")) # gives the
         legend lines the correct color and width


####### Observed Values #####
F_observed<-c(rep(0,n))
m=n+0.25
for (i in 1:n)
{
  F_observed[i]<-(i-0.375)/m
}

###### Function to compute SS ######
SS<-function(F,F_line,n){
  SS<-0
  for (i in 1:n)
  {
    SS<-SS+(F[i]-F_line[i])^2
  }
  return(SS)
}

####### SS values for the full model and submodels #######
SS1<-SS(F1,F_observed,n)
SS1
SS2<-SS(F2,F_observed,n)
SS2
SS3<-SS(F3,F_observed,n)
SS3
SS4<-SS(F4,F_observed,n)
```

SS4

```
#####################################
op <- par(mfrow=c(1,1))
plot(F_observed,F_observed,type='l',xlab="Observed probability(carbon
   fibre Data)",ylab="Expected probability")
lines(F_observed,F1,col='blue',type='l')
lines(F_observed,F2,col='red')
lines(F_observed,F3,col='purple',type="l")
lines(F_observed,F4,col='black',type="l")

legend(0.6,0.4,pt.cex = 1,cex=0.6, # places a legend at the appropriate
   place
      c("MOLLoGETE",expression(MOLLEGETE(paste(beta,",",delta,",",1,",",
         lambda))),expression(MOLLEGETE(paste(1,",",delta,",",c,",",
         lambda))),"GD"), # puts text in the legend
      lty=c(1,1,1,1), # gives the legend appropriate symbols (lines)
      lwd=c(1,1,1,1),col=c("blue","red","purple","black")) # gives the
         legend lines the correct color and width

###### Parameter estimation and goodness-of-fit statistics of the model
   and sub-models#######

#### loading the adequacyModel library #########
library(AdequacyModel)

#################################
###Carbon Fibre Data########
data=c(3.70, 2.74, 2.73, 2.50, 3.60, 3.11, 3.27,

      2.87, 1.47, 3.11, 4.42, 2.41, 3.19, 3.22, 1.69,3.28, 3.09, 1.87,
         3.15, 4.90, 3.75,

      2.43, 2.95, 2.97, 3.39, 2.96, 2.53, 2.67, 2.93, 3.22, 3.39, 2.81,
         4.20, 3.33, 2.55,

      3.31, 3.31, 2.85, 2.56, 3.56, 3.15, 2.35, 2.55, 2.59, 2.38, 2.81,
         2.77, 2.17, 2.83,

      1.92, 1.41, 3.68, 2.97, 1.36, 0.98, 2.76, 4.91, 3.68, 1.84, 1.59,
         3.19, 1.57, 0.81,

      5.56, 1.73, 1.59, 2.00, 1.22, 1.12, 1.71, 2.17, 1.17, 5.08, 2.48,
         1.18, 3.51, 2.17,
```

```
      1.69, 1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.70,
         2.03, 1.80, 1.57,

      1.08, 2.03, 1.61, 2.12, 1.89, 2.88, 2.82, 2.05, 3.65)

##### Marshall olkin Log Logistic Erlang Truncated Extended (MOLLoGETE)
   ########
##### PDF #######
MOLLoGETE_pdf<-function(par,x){
  beta=par[1]
  delta=par[2]
  c=par[3]
  lambda=par[4]

  (exp(-beta*(1-exp(-lambda))*x)*(1+x^c)^(-2)*delta*(c*x^(c-1)+(1+x^c)*
     beta*(1-exp(-lambda))))/(1-(1-delta)*exp(-beta*(1-exp(-lambda))*x)
     *(1+x^c)^(-1))^2
}
####### CDF #######
MOLLoGETE_cdf<-function(par,x){
  beta=par[1]
  delta=par[2]
  c=par[3]
  lambda=par[4]

  (1-(1+x^c)^(-1)*exp(-beta*(1-exp(-lambda))*x))/(1-(1-delta)*(1+x^c)
     ^(-1)*exp(-beta*(1-exp(-lambda))*x))
}
#### Goodness-of-fit statistics #######
goodness.fit(pdf=MOLLoGETE_pdf, cdf=MOLLoGETE_cdf,starts=c(1,1,1,1),data
    = data,method = "BFGS",
           domain = c(0,Inf),mle = NULL, lim_inf = c(0,0,0,0),
           lim_sup = c(Inf,Inf,Inf,Inf))

############# MOLLoGETE sub-model with c=1 #################
######## PDF ##########
MOLLoGETE1_pdf<-function(par,x){
  beta=par[1]
  delta=par[2]
  c=1
  lambda=par[3]

  (exp(-beta*(1-exp(-lambda))*x)*(1+x^c)^(-2)*delta*(c*x^(c-1)+(1+x^c)*
```

```
      beta*(1-exp(-lambda))))/(1-(1-delta)*exp(-beta*(1-exp(-lambda))*x)
      *(1+x^c)^(-1))^2
}
#### CDF #######
MOLLoGETE1_cdf<-function(par,x){
  beta=par[1]
  delta=par[2]
  c=1
  lambda=par[3]

  (1-(1+x^c)^(-1)*exp(-beta*(1-exp(-lambda))*x))/(1-(1-delta)*(1+x^c)
      ^(-1)*exp(-beta*(1-exp(-lambda))*x))
}
#### Goodness-of-fit statistics #######
goodness.fit(pdf=MOLLoGETE1_pdf, cdf=MOLLoGETE1_cdf,starts=c
    (2.9,.05,2.9),data = data,method = "C",
             domain = c(0,Inf),mle = NULL, lim_inf = c(0,0,0),
             lim_sup = c(Inf,Inf,Inf))

############# MOLLoGETE sub-model with beta=0.00001 ##################
######## PDF ##########
MOLLoGETE2_pdf<-function(par,x){
  beta=1
  delta=par[1]
  c=par[2]
  lambda=par[3]

  (exp(-beta*(1-exp(-lambda))*x)*(1+x^c)^(-2)*delta*(c*x^(c-1)+(1+x^c)*
      beta*(1-exp(-lambda))))/(1-(1-delta)*exp(-beta*(1-exp(-lambda))*x)
      *(1+x^c)^(-1))^2
}
#### CDF #######
MOLLoGETE2_cdf<-function(par,x){
  beta=1
  delta=par[1]
  c=par[2]
  lambda=par[3]

  (1-(1+x^c)^(-1)*exp(-beta*(1-exp(-lambda))*x))/(1-(1-delta)*(1+x^c)
      ^(-1)*exp(-beta*(1-exp(-lambda))*x))
}
#### Goodness-of-fit statistics #######
goodness.fit(pdf=MOLLoGETE2_pdf, cdf=MOLLoGETE2_cdf,starts=c(2,1,1),data
     = data,method = "C",
```

```
            domain = c(0,Inf),mle = NULL, lim_inf = c(0,0,0),
            lim_sup = c(Inf,Inf,Inf))

#############Sub-models with beta->0 and c=1 #########
######## PDF ##########
MOLLoGETE3_pdf<-function(par,x){
  beta=0.00001
  delta=par[1]
  c=1
  lambda=par[2]

  (exp(-beta*(1-exp(-lambda))*x)*(1+x^c)^(-2)*delta*(c*x^(c-1)+(1+x^c)*
     beta*(1-exp(-lambda))))/(1-(1-delta)*exp(-beta*(1-exp(-lambda))*x)
     *(1+x^c)^(-1))^2
}
#### CDF #######
MOLLoGETE3_cdf<-function(par,x){
  beta=0.00001
  delta=par[1]
  c=1
  lambda=par[2]

  (1-(1+x^c)^(-1)*exp(-beta*(1-exp(-lambda))*x))/(1-(1-delta)*(1+x^c)
     ^(-1)*exp(-beta*(1-exp(-lambda))*x))
}
#### Goodness-of-fit statistics #######
goodness.fit(pdf=MOLLoGETE3_pdf, cdf=MOLLoGETE3_cdf,starts=c(.1,0.1),
    data = data,method = "C",
            domain = c(0,Inf),mle = NULL, lim_inf = c(0,0),
            lim_sup = c(Inf,Inf))



#########Non Nested Gamma-Dagum model ###################
GD_pdf<-function(par,x){
  alpha=par[1]
  beta=par[2]
  delta= par[3]
  lambda= par[4]
  (1/gamma(alpha))*beta*delta*lambda*x^(-delta-1)*(1+lambda*x^(-delta))
     ^(-beta-1)*(-log(1-(1+lambda*x^(-delta))^(-beta)))^(alpha-1)
}

GD_cdf<-function(par,x){
```

```
  alpha=par[1]
  beta=par[2]
  delta= par[3]
  lambda= par[4]

  pgamma(-log(1-(1+lambda*x^(-delta))^(-beta)),alpha)
}



goodness.fit(pdf=GD_pdf, cdf=GD_cdf,starts=c(1,1,1,1),data = data,method
     = "BFGS",
            domain = c(0,Inf),mle = NULL, lim_inf = c(0,0,0,0),
            lim_sup = c(Inf,Inf,Inf,Inf))
```

## Application 2

Waiting Times Data

```
require(gsl)
require(stats)

###Customers Waiting Times#######
data=c(0.8,0.8,1.3,1.5,1.8,1.9,1.9,2.1,2.6,2.7,
       2.9,3.1,3.2,3.3,3.5,3.6,4.0,4.1,4.2,4.2,
       4.3,4.3,4.4,4.4,4.6,4.7,4.7,4.8,4.9,4.9,
       5.0,5.3,5.5,5.7,5.7,6.1,6.2,6.2,6.2,6.3,
       6.7,6.9,7.1,7.1,7.1,7.1,7.4,7.6,7.7,8.0,
       8.2,8.6,8.6,8.6,8.8,8.8,8.9,8.9,9.5,9.6,
       9.7,9.8,10.7,10.9,11.0,11.0,11.1,11.2,11.2,11.5,
       11.9,12.4,12.5,12.9,13.0,13.1,13.3,13.6,13.7,13.9,
       14.1,15.4,15.4,17.3,17.3,18.1,18.2,18.4,18.9,19.0,
       19.9,20.6,21.3,21.4,21.9,23.0,27.0,31.6,33.1,38.5)
data<-sort(data)
n=length(data)

####### Full MOLLEGETE Model ########
###### Parameter estimates ######
beta1=0.76157344
delta1=59.99701639
c1= 1.69393367
lambda1= 0.09269573
###### PDF #########
MOLLOGETE_pdf <- function(data,beta,delta,c,lambda,n) {
  f<-c(rep(0,n))
  for (i in 1:n)
  {
    x<-data[i]
    f[i]<-(exp(-beta*(1-exp(-lambda))*x)*(1+x^c)^(-2)*delta*(c*x^(c-1)
        +(1+x^c)*beta*(1-exp(-lambda))))/(1-(1-delta)*exp(-beta*(1-exp(-
        lambda))*x)*(1+x^c)^(-1))^2
  }
  return(f)
}
###### FIT ###
f1<-MOLLOGETE_pdf(data,beta1,delta1,c1,lambda1,n)

######### CDF #########
MOLLOGETE_cdf <- function(data,beta,delta,c,lambda,n){
  F<-c(rep(0,n))
```

```
  for (i in 1:n)
  {
    x<-data[i]
    F[i]<-(1-(1+x^c)^(-1)*exp(-beta*(1-exp(-lambda))*x))/(1-(1-delta)*(1+
        x^c)^(-1)*exp(-beta*(1-exp(-lambda))*x))
  }
  return(F)
}
########
F1<-MOLLOGETE_cdf(data,beta1,delta1,c1,lambda1,n)
F1

############# Sub model with c=1 ################
#####Parameter estimates#######
beta2 = 1.46464258
delta2=4.73977985
lambda2=0.02972164

####### PDF #########
MOLLOGETE1_pdf <- function(data,beta,delta,lambda,n) {
  c=1
  f<-c(rep(0,n))
  for (i in 1:n)
  {
    x<-data[i]
    f[i]<-(exp(-beta*(1-exp(-lambda))*x)*(1+x^c)^(-2)*delta*(c*x^(c-1)
        +(1+x^c)*beta*(1-exp(-lambda))))/(1-(1-delta)*exp(-beta*(1-exp(-
        lambda))*x)*(1+x^c)^(-1))^2
  }
  return(f)
}
####### FITS #########
f2<-MOLLOGETE1_pdf(data,beta2,delta2,lambda2,n)

######### CDF #########
MOLLOGETE1_cdf <- function(data,beta,delta,lambda,n){
  c=1
  F<-c(rep(0,n))
  for (i in 1:n)
  {
    x<-data[i]
    F[i]<-(1-(1+x^c)^(-1)*exp(-beta*(1-exp(-lambda))*x))/(1-(1-delta)*(1+
        x^c)^(-1)*exp(-beta*(1-exp(-lambda))*x))
  }
```

```
  return(F)
}
########
F2<-MOLLOGETE1_cdf(data,beta2,delta2,lambda2,n)
F2


################# Sub Model with beta=1 ###############
###### Parameter estimates #####
delta3=3.63978341
c3=0.06208966
lambda3= 0.16346091
####### PDF #########
MOLLOGETE2_pdf <- function(data,delta,c,lambda,n) {
  beta=1
  f<-c(rep(0,n))
  for (i in 1:n)
  {
    x<-data[i]
    f[i]<-(exp(-beta*(1-exp(-lambda))*x)*(1+x^c)^(-2)*delta*(c*x^(c-1)
        +(1+x^c)*beta*(1-exp(-lambda))))/(1-(1-delta)*exp(-beta*(1-exp(-
        lambda))*x)*(1+x^c)^(-1))^2
  }
  return(f)
}
####### FITS #########
f3<-MOLLOGETE2_pdf(data,delta3,c3,lambda3,n)

######### CDF #########
MOLLOGETE2_cdf <- function(data,delta,c,lambda,n){
  beta=1
  F<-c(rep(0,n))
  for (i in 1:n)
  {
    x<-data[i]
    F[i]<-(1-(1+x^c)^(-1)*exp(-beta*(1-exp(-lambda))*x))/(1-(1-delta)*(1+
        x^c)^(-1)*exp(-beta*(1-exp(-lambda))*x))
  }
  return(F)
}
########
F3<-MOLLOGETE2_cdf(data,delta3,c3,lambda3,n)
F3
```

```
#############Sub-models with beta->0 and c=1 #########
###### Parameter estimates #####
delta5=7.6555995
lambda5=0.7549982
####### PDF #########
MOLLOGETE4_pdf <- function(data,delta,lambda,n) {
  beta=0.00001
  c=1
  f<-c(rep(0,n))
  for (i in 1:n)
  {
    x<-data[i]
    f[i]<-(exp(-beta*(1-exp(-lambda))*x)*(1+x^c)^(-2)*delta*(c*x^(c-1)
        +(1+x^c)*beta*(1-exp(-lambda))))/(1-(1-delta)*exp(-beta*(1-exp(-
        lambda))*x)*(1+x^c)^(-1))^2
  }
  return(f)
}
####### FITS #########
f5<-MOLLOGETE4_pdf(data,delta5,lambda5,n)

######### CDF #########
MOLLOGETE4_cdf <- function(data,delta,lambda,n){
  beta=0.00001
  c=1
  F<-c(rep(0,n))
  for (i in 1:n)
  {
    x<-data[i]
    F[i]<-(1-(1+x^c)^(-1)*exp(-beta*(1-exp(-lambda))*x))/(1-(1-delta)*(1+
        x^c)^(-1)*exp(-beta*(1-exp(-lambda))*x))
  }
  return(F)
}
########
F5<-MOLLOGETE4_cdf(data,delta5,lambda5,n)
F5
```

```
########## Non-nested Gamma Dagum model################
###### Parameter estimates ##### & & &
alpha4=0.1842851
beta4=8.1741168
delta4=1.7438406
lambda4= 22.9712633
####### PDF #########
GD_pdf <- function(data,alpha,beta,delta,lambda,n) {
   f<-c(rep(0,n))
  for (i in 1:n)
  {
    x<-data[i]
    f[i]<-(1/gamma(alpha))*beta*delta*lambda*x^(-delta-1)*(1+lambda*x^(-
        delta))^(-beta-1)*(-log(1-(1+lambda*x^(-delta))^(-beta)))^(alpha
        -1)
  }
  return(f)
}
####### FITS #########
f4<-GD_pdf(data,alpha4,beta4,delta4,lambda4,n)

######### CDF #########
GD_cdf <- function(data,alpha,beta,delta,lambda,n){
  F<-c(rep(0,n))
  for (i in 1:n)
  {
    x<-data[i]
    F[i]<-pgamma(-log(1-(1+lambda*x^(-delta))^(-beta)),alpha)
  }
  return(F)
}
########
F4<-GD_cdf(data,alpha4,beta4,delta4,lambda4,n)
F4

####### MOGETTE DISTRIBUTION ###########################
###### Parameter estimates ##### & & &
alpha5=4.1156088
beta5=0.8399391
lambda5 = 0.2600705

####### PDF #########

##### PDF #######
```

```
MOGETE_pdf<-function(data, alpha, beta ,lambda, n){
  F<-c(rep(0,n))
  for (i in 1:n)
  {
    x<-data[i]
    F[i]<- alpha*beta*(1-exp(-lambda))*exp(-beta*(1-exp(-lambda))*x)
        *(1-(1-alpha)*exp(-beta*(1-exp(-lambda))*x))^(-2)
  }
  return(F)


}
####### CDF #######
MOGETE_cdf<-function(data, alpha, beta ,lambda, n){
  F<-c(rep(0,n))
  for (i in 1:n)
  {
    x<-data[i]
    F[i]<- (1-exp(-beta*(1-exp(-lambda))*x))/(1-(1-alpha)*exp(-beta*(1-
        exp(-lambda))*x))
  }
  return(F)
}
f5<-MOGETE_pdf(data,alpha5,beta5,lambda5,n)
F5<-MOGETE_cdf(data,alpha5,beta5,lambda5,n)



######### FItting the densities of the models #################
op <- par(mfrow=c(1,1))
X <- data
hist(X, prob=TRUE,ylim=c(0,0.15),main=' ', breaks = 20) # prob=TRUE for
    probabilities not counts # add a density estimate with defaults
lines(data,f1,col='blue',type='l')
#lines(data,f2,col='red')
#lines(data,f3,col='purple',type='l')
lines(data,f4,col='black',type='l')

legend(30,0.1,pt.cex = 1,cex=0.6, # places a legend at the appropriate
    place
      c("MOLLoGETE","GD"), # puts text in the legend
      lty=c(1,1), # gives the legend appropriate symbols (lines)
      lwd=c(1,1),col=c("blue","black")) # gives the legend lines the
          correct color and width
```

```
####### Observed Values #####
F_observed<-c(rep(0,n))
m=n+0.25
for (i in 1:n)
{
  F_observed[i]<-(i-0.375)/m
}

###### Function to compute SS ######
SS<-function(F,F_line,n){
  SS<-0
  for (i in 1:n)
  {
    SS<-SS+(F[i]-F_line[i])^2
  }
  return(SS)
}


####### SS values for the full model and submodels #######
SS1<-SS(F1,F_observed,n)
SS1
SS2<-SS(F2,F_observed,n)
SS2
SS3<-SS(F3,F_observed,n)
SS3
SS4<-SS(F4,F_observed,n)
SS4
SS5<-SS(F5,F_observed,n)
SS5

######################################
op <- par(mfrow=c(1,1))
plot(F_observed,F_observed,type='l',xlab="Observed probability(Customers
     Waiting times Data)",ylab="Expected probability")
lines(F_observed,F1,col='blue',type='l')
#lines(F_observed,F2,col='red')
#lines(F_observed,F3,col='purple',type="l")
lines(F_observed,F4,col='black',type="l")

legend(0.6,0.4,pt.cex = 1,cex=0.6, # places a legend at the appropriate
    place
```

```
      c("MOLLoGETE","GD"), # puts text in the legend
      lty=c(1,1), # gives the legend appropriate symbols (lines)
      lwd=c(1,1),col=c("blue","black")) # gives the legend lines the
         correct color and width

###### Parameter estimation and goodness-of-fit Statistics of the model
   and sub-models#######

#### loading the adequacyModel library #########
library(AdequacyModel)

#################################
### Customers Waiting Times Data ########
data=c(0.8,0.8,1.3,1.5,1.8,1.9,1.9,2.1,2.6,2.7,
      2.9,3.1,3.2,3.3,3.5,3.6,4.0,4.1,4.2,4.2,
      4.3,4.3,4.4,4.4,4.6,4.7,4.7,4.8,4.9,4.9,
      5.0,5.3,5.5,5.7,5.7,6.1,6.2,6.2,6.2,6.3,
      6.7,6.9,7.1,7.1,7.1,7.1,7.4,7.6,7.7,8.0,
      8.2,8.6,8.6,8.6,8.8,8.8,8.9,8.9,9.5,9.6,
      9.7,9.8,10.7,10.9,11.0,11.0,11.1,11.2,11.2,11.5,
      11.9,12.4,12.5,12.9,13.0,13.1,13.3,13.6,13.7,13.9,
      14.1,15.4,15.4,17.3,17.3,18.1,18.2,18.4,18.9,19.0,
      19.9,20.6,21.3,21.4,21.9,23.0,27.0,31.6,33.1,38.5

)

##### Marshall olkin Log Logistic Erlang Truncated Extended (MOLLoGETE)
   ########
##### PDF #######
MOLLoGETE_pdf<-function(par,x){
  beta=par[1]
  delta=par[2]
  c=par[3]
  lambda=par[4]

  (exp(-beta*(1-exp(-lambda))*x)*(1+x^c)^(-2)*delta*(c*x^(c-1)+(1+x^c)*
     beta*(1-exp(-lambda))))/(1-(1-delta)*exp(-beta*(1-exp(-lambda))*x)
     *(1+x^c)^(-1))^2
}
####### CDF #######
MOLLoGETE_cdf<-function(par,x){
  beta=par[1]
  delta=par[2]
  c=par[3]
```

```
  lambda=par[4]

  (1-(1+x^c)^(-1)*exp(-beta*(1-exp(-lambda))*x))/(1-(1-delta)*(1+x^c)
     ^(-1)*exp(-beta*(1-exp(-lambda))*x))
}
#### Goodness-of-fit statistics ######
goodness.fit(pdf=MOLLoGETE_pdf, cdf=MOLLoGETE_cdf,starts=c(.8,1,1,.2),
   data = data,method = "BFGS",
           domain = c(0,Inf),mle = NULL, lim_inf = c(0,0,0,0),
           lim_sup = c(Inf,Inf,Inf,Inf))

############# MOLLoGETE sub-model with c=1 #################
######## PDF ##########
MOLLoGETE1_pdf<-function(par,x){
  beta=par[1]
  delta=par[2]
  c=1
  lambda=par[3]

  (exp(-beta*(1-exp(-lambda))*x)*(1+x^c)^(-2)*delta*(c*x^(c-1)+(1+x^c)*
     beta*(1-exp(-lambda))))/(1-(1-delta)*exp(-beta*(1-exp(-lambda))*x)
     *(1+x^c)^(-1))^2
}
#### CDF #######
MOLLoGETE1_cdf<-function(par,x){
  beta=par[1]
  delta=par[2]
  c=1
  lambda=par[3]

  (1-(1+x^c)^(-1)*exp(-beta*(1-exp(-lambda))*x))/(1-(1-delta)*(1+x^c)
     ^(-1)*exp(-beta*(1-exp(-lambda))*x))
}
#### Goodness-of-fit statistics #######
goodness.fit(pdf=MOLLoGETE1_pdf, cdf=MOLLoGETE1_cdf,starts=c(2,.3,.9),
   data = data,method = "C",
           domain = c(0,Inf),mle = NULL, lim_inf = c(0,0,0),
           lim_sup = c(Inf,Inf,Inf))

############# MOLLoGETE sub-model with beta=0.00001 #################
######## PDF ##########
MOLLoGETE2_pdf<-function(par,x){
  beta=1
  delta=par[1]
```

```
  c=par[2]
  lambda=par[3]

  (exp(-beta*(1-exp(-lambda))*x)*(1+x^c)^(-2)*delta*(c*x^(c-1)+(1+x^c)*
      beta*(1-exp(-lambda))))/(1-(1-delta)*exp(-beta*(1-exp(-lambda))*x)
      *(1+x^c)^(-1))^2
}
#### CDF #######
MOLLoGETE2_cdf<-function(par,x){
  beta=1
  delta=par[1]
  c=par[2]
  lambda=par[3]

  (1-(1+x^c)^(-1)*exp(-beta*(1-exp(-lambda))*x))/(1-(1-delta)*(1+x^c)
      ^(-1)*exp(-beta*(1-exp(-lambda))*x))
}
#### Goodness-of-fit statistics #######
goodness.fit(pdf=MOLLoGETE2_pdf, cdf=MOLLoGETE2_cdf,starts=c(2,1,1),data
     = data,method = "C",
            domain = c(0,Inf),mle = NULL, lim_inf = c(0,0,0),
            lim_sup = c(Inf,Inf,Inf))

#############Sub-models with beta->0 and c=1 #########
######## PDF ##########
MOLLoGETE3_pdf<-function(par,x){
  beta=0.00001
  delta=par[1]
  c=1
  lambda=par[2]

  (exp(-beta*(1-exp(-lambda))*x)*(1+x^c)^(-2)*delta*(c*x^(c-1)+(1+x^c)*
      beta*(1-exp(-lambda))))/(1-(1-delta)*exp(-beta*(1-exp(-lambda))*x)
      *(1+x^c)^(-1))^2
}
#### CDF #######
MOLLoGETE3_cdf<-function(par,x){
  beta=0.00001
  delta=par[1]
  c=1
  lambda=par[2]

  (1-(1+x^c)^(-1)*exp(-beta*(1-exp(-lambda))*x))/(1-(1-delta)*(1+x^c)
      ^(-1)*exp(-beta*(1-exp(-lambda))*x))
```

```
}
#### Goodness-of-fit statistics #######
goodness.fit(pdf=MOLLoGETE3_pdf, cdf=MOLLoGETE3_cdf,starts=c(10,.001),
    data = data,method = "C",
            domain = c(0,Inf),mle = NULL, lim_inf = c(0,0),
            lim_sup = c(Inf,Inf))



#########Non Nested Gamma-Dagum model ####################
GD_pdf<-function(par,x){
  alpha=par[1]
  beta=par[2]
  delta= par[3]
  lambda= par[4]
  (1/gamma(alpha))*beta*delta*lambda*x^(-delta-1)*(1+lambda*x^(-delta))
     ^(-beta-1)*(-log(1-(1+lambda*x^(-delta))^(-beta)))^(alpha-1)
}

GD_cdf<-function(par,x){
  alpha=par[1]
  beta=par[2]
  delta= par[3]
  lambda= par[4]

  pgamma(-log(1-(1+lambda*x^(-delta))^(-beta)),alpha)
}

goodness.fit(pdf=GD_pdf, cdf=GD_cdf,starts=c(1,1,1,1),data = data,method
     = "BFGS",
            domain = c(0,Inf),mle = NULL, lim_inf = c(0,0,0,0),
            lim_sup = c(Inf,Inf,Inf,Inf))

##### MOGETE DISTRIBUTION ########
##### PDF #######
MOGETE_pdf<-function(par,x){
  alpha=par[1]
  beta=par[2]
  lambda=par[3]

  alpha*beta*(1-exp(-lambda))*exp(-beta*(1-exp(-lambda))*x)*(1-(1-alpha)
     *exp(-beta*(1-exp(-lambda))*x))^(-2)
}
####### CDF #######
```

```
MOGETE_cdf<-function(par,x){
  alpha=par[1]
  beta=par[2]
  lambda=par[3]

  (1-exp(-beta*(1-exp(-lambda))*x))/(1-(1-alpha)*exp(-beta*(1-exp(-
      lambda))*x))
}
#### Goodness-of-fit statistics ######
goodness.fit(pdf=MOGETE_pdf, cdf=MOGETE_cdf,starts=c(.8,1,.2),data =
    data,method = "BFGS",
            domain = c(0,Inf),mle = NULL, lim_inf = c(0,0,0),
            lim_sup = c(Inf,Inf,Inf))
```

### Plots of pdf and hazard function

```
# Marshall Olkin Log-logistic Erlang Truncated Distribution
rm(list=ls())
library(stats4)
library(bbmle)
library(stats)

###############Define Function###################
#define MOLLOGETE cdf
MOLLOGETE_cdf <- function(beta, delta, c, lambda, x ){

  (1-(1+x^c)^(-1)*exp(-beta*(1-exp(-lambda))*x))/(1-(1-delta)*(1+x^c)
      ^(-1)*exp(-beta*(1-exp(-lambda))*x))

}

#define MOLLOGETE pdf
MOLLOGETE_pdf <- function(beta, delta, c, lambda, x){
  (exp(-beta*(1-exp(-lambda))*x)*(1+x^c)^(-2)*delta*(c*x^(c-1)+(1+x^c)*
      beta*(1-exp(-lambda))))/(1-(1-delta)*exp(-beta*(1-exp(-lambda))*x)
      *(1+x^c)^(-1))^2

}

# check
f=function(x){MOLLOGETE_pdf(1,5,4,3,x)}
integrate(f,0,4)$value-MOLLOGETE_cdf(1,5,4,3,4)



#define MOLLOGETE Quantiles
MOLLOGETE_quantile <- function(beta, delta, c, lambda, u){
  f <- function(x){
   log((1-u)/(delta+(1-u)*(1-delta)))+beta*(1-exp(-lambda))*x+log(1+x^c)
  }
  rc <- uniroot(f, lower = 0, upper = 100, tol = 1e-9)
  result <- rc$root
  #error <-MOLLOGETE_cdf(beta, delta, c, lambda, result)-u
  return(result)
  #return(list("result"=result, "error"=error))
}
#Get quantiles of selected parameter values
y<-c(0,0,0,0,0,0,0,0,0)
```

```
for (i in 1:9){
 y[i]<- MOLLOGETE_quantile(2,5,6,7,i*.1)
}
# define MOLLW hazard
# c,alpha,beta,delta>0
#define MOLLOGETE Hazard
MOLLOGETE_hazard <- function(beta, delta, c, lambda, x){
  MOLLOGETE_pdf(beta, delta, c, lambda, x) /
    (1 - MOLLOGETE_cdf(beta, delta, c, lambda, x))
}

#plots of pdf functions
x=seq(0,3,by=0.001)
z1=MOLLOGETE_pdf(0.2,0.4,1,0.2,x)
plot(x,z1,ylim=c(0,4),col=2,'l',lwd=2,xlab="x",ylab="g(x)")

z2=MOLLOGETE_pdf(.5,0.01,1,0.01,x)
lines(x,z2,col=3,lwd=2)

z3=MOLLOGETE_pdf(3.8,8,5.0,1000,x)
lines(x,z3,col=4,lwd=2)

z4=MOLLOGETE_pdf(0.1,2.5,4.5,1000,x)
lines(x,z4,col=5,lwd=2)

legend(1.5, 4,pt.cex = 0.9,cex=0.4,c(
  expression(paste(beta,'=0.20,',delta,'=0.40,',c,'=1.00,',lambda
      ,'=0.20')),
  expression(paste(beta,'=0.50,',delta,'=0.01,',c,'=1.00,',lambda
      ,'=0.01')),
  expression(paste(beta,'=3.80,',delta,'=8.00,',c,'=5.00,',lambda
      ,'=1000')),
  expression(paste(beta,'=0.10,',delta,'=2.50,',c,'=4.50,',lambda
      ,'=1000'))),col=c(2,3,4,5),lwd=c(1,1,1,1))
x=seq(0,3,by=0.001)
z5=MOLLOGETE_pdf(4.0,4.8,8.9,3.5,x)
plot(x,z5,ylim=c(0,5),col=3,'l',lwd=2,xlab="x",ylab="g(x)")

z6=MOLLOGETE_pdf(6.5,2.7,7.5,3,x)
lines(x,z6,col=4,lwd=2)


z7=MOLLOGETE_pdf(0.8,0.5,3.5,0.2,x)
lines(x,z7,col=5,lwd=2)
```

```
z8=MOLLOGETE_pdf(.5,.1,.1,.1,x)
lines(x,z8,col=6,lwd=2)

legend(1.0, 5.0,pt.cex = .9,cex=0.4,c(
  expression(paste(beta,'=4.0,',delta,'=4.8,',c,'=8.9,',lambda,'=3.5')),
  expression(paste(beta,'=6.5,',delta,'=2.7,',c,'=7.5,',lambda,'=3.0')),
  expression(paste(beta,'=0.8,',delta,'=0.5,',c,'=3.5,',lambda,'=0.2')),
  expression(paste(beta,'=0.5,',delta,'=0.1,',c,'=0.1,',lambda,'=0.1')))
      ,col=c(3,4,5,6),lwd=c(1,1,1,1))

#plots of Hazard functions
x=seq(0,3,by=0.001)
y1=MOLLOGETE_hazard(0.2,0.4,1,0.2,x)
plot(x,y1,ylim=c(0,10),col=2,'l',lwd=2,xlab="x",ylab="h(x)")

y2=MOLLOGETE_hazard(5,0.1,1,0.01,x)
lines(x,y2,col=3,lwd=2)

y3=MOLLOGETE_hazard(3.8,0.5,5.0,1000,x)
lines(x,y3,col=4,lwd=2)

y4=MOLLOGETE_hazard(0.1,2.5,4.5,1000,x)
lines(x,y4,col=5,lwd=2)

legend(1.5, 10,pt.cex = 0.9,cex=0.4,c(
  expression(paste(beta,'=0.20,',delta,'=0.40,',c,'=1.00,',lambda
      ,'=0.20')),
  expression(paste(beta,'=5.00,',delta,'=0.10,',c,'=1.00,',lambda
      ,'=0.01')),
  expression(paste(beta,'=3.80,',delta,'=0.50,',c,'=5.00,',lambda
      ,'=1000')),
  expression(paste(beta,'=0.10,',delta,'=2.50,',c,'=4.50,',lambda
      ,'=1000'))),col=c(2,3,4,5),lwd=c(1,1,1,1))

x=seq(0,3,by=0.001)
y5=MOLLOGETE_hazard(4.0,0.8,7.9,3.5,x)
plot(x,y5,ylim=c(0,10),col=3,'l',lwd=2,xlab="x",ylab="h(x)")

y6=MOLLOGETE_hazard(6.5,2.7,0.7,0.05,x)
lines(x,y6,col=4,lwd=2)


y7=MOLLOGETE_hazard(2.8,0.5,3.5,0.2,x)
```

```
lines(x,y7,col=5,lwd=2)

y8=MOLLOGETE_hazard(0.5,.1,.1,.2,x)
lines(x,y8,col=6,lwd=2)

legend(1.5, 6.0,pt.cex = 0.9,cex=0.4,c(
  expression(paste(beta,'=4.00,',delta,'=0.80,',c,'=7.90,',lambda
      ,'=3.50')),
  expression(paste(beta,'=6.50,',delta,'=2.70,',c,'=0.70,',lambda
      ,'=0.05')),
  expression(paste(beta,'=2.80,',delta,'=0.50,',c,'=3.50,',lambda
      ,'=0.20')),
  expression(paste(beta,'=0.50,',delta,'=0.10,',c,'=0.10,',lambda
      ,'=0.20'))),col=c(3,4,5,6),lwd=c(1,1,1,1))
```

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Table 2: Monte Carlo Simulation Results: Average Bias and RMSE

| | | I | | II | |
|---|---|---|---|---|---|
| Parameter | n | Average Bias | RMSE | Average Bias | RMSE |
| $\beta$ | 30 | -0.00173 | 0.00488 | 0.13106 | 0.37350 |
| | 50 | -0.00158 | 0.00461 | 0.11955 | 0.37116 |
| | 100 | -0.00133 | 0.00456 | 0.08055 | 0.30743 |
| | 200 | -0.00119 | 0.00412 | 0.07242 | 0.30691 |
| | 400 | -0.00096 | 0.00394 | 0.07099 | 0.30479 |
| | 800 | -0.00076 | 0.00371 | 0.05272 | 0.30324 |
| $\delta$ | 30 | 0.00311 | 0.06970 | 0.28189 | 0.62299 |
| | 50 | 0.00261 | 0.06516 | 0.25945 | 0.60542 |
| | 100 | 0.00204 | 0.06273 | 0.18003 | 0.48055 |
| | 200 | 0.00197 | 0.05647 | 0.08588 | 0.47334 |
| | 400 | 0.00147 | 0.05408 | 0.07699 | 0.44456 |
| | 800 | 0.00121 | 0.05284 | 0.05539 | 0.43979 |
| c | 30 | 0.00636 | 0.03237 | -0.19197 | 0.49129 |
| | 50 | 0.00621 | 0.03148 | -0.15932 | 0.44081 |
| | 100 | 0.00550 | 0.03057 | -0.12329 | 0.42227 |
| | 200 | 0.00408 | 0.02935 | -0.03853 | 0.38092 |
| | 400 | 0.00331 | 0.02169 | -0.02038 | 0.34313 |
| | 800 | 0.00282 | 0.01960 | -0.01902 | 0.33272 |
| $\lambda$ | 30 | 0.00320 | 0.02028 | -0.00080 | 0.00176 |
| | 50 | 0.00246 | 0.01615 | -0.00052 | 0.00119 |
| | 100 | 0.00146 | 0.01573 | -0.00029 | 0.00094 |
| | 200 | 0.00117 | 0.01505 | -0.00018 | 0.00074 |
| | 400 | 0.00102 | 0.01444 | -0.00013 | 0.00063 |
| | 800 | 0.00094 | 0.01403 | -0.00011 | 0.00058 |

Table 3: Parameter estimates, log-likelihood, AIC, AICC, BIC, $W^*$ and $A^*$

| | Estimates | | | | Statistics | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | $\beta$ | $\delta$ | c | $\lambda$ | $-2\log L$ | AIC | AICC | BIC | W* | A* | SS |
| MOLLoGETE | 1.6003 | 114.4066 | 1.4990 | 1.4149 | 283.9276 | 291.9276 | 292.3487 | 302.3483 | 0.0768 | 0.4371 | 0.0714 |
| | (7.1761) | (63.6599) | (1.2092) | (13.9366) | | | | | | | |
| MOLLoGETE(1,$\delta$,$c$,$\lambda$) | 1 | 17.3095 | 3.1216 | 0.1080 | 299.1980 | 305.2088 | 305.4588 | 313.0243 | 0.2353 | 1.2185 | 0.5806 |
| | - | (4.3173) | (0.5629) | (0.2480) | | | | | | | |
| MOLLoGETE($\beta$,$\delta$,1,$\lambda$) | 0.8532 | 18.0307 | 1 | 2.5780 | 310.5560 | 316.5644 | 316.8144 | 324.3799 | 0.0797 | 0.4587 | 1.0340 |
| | (0.2554) | (3.7521) | - | (3.4538) | | | | | | | |
| MOLLoGETE(0,$\delta$,1,$\lambda$ ) | 0.00001 | 2.4192 | 1 | 0.3452 | 462.2412 | 466.2412 | 466.3649 | 471.4516 | 0.2704 | 1.4381 | 3.6479 |
| | - | (0.3498) | - | (24.1622) | | | | | | | |
| LLoG | - | - | 1.6287 | - | 467.6428 | 469.6428 | 469.6836 | 472.2479 | 0.3645 | 1.9393 | 11.6714 |
| | - | - | (0.1288) | - | | | | | | | |
| | $\alpha$ | $\beta$ | $\delta$ | $\lambda$ | | | | | | | |
| GD | 0.2172 | 5.3891 | 3.4171 | 20.0139 | 291.6666 | 299.6666 | 300.0876 | 310.0873 | 0.2255 | 1.1703 | 0.2176 |
| | (0.0211) | (0.0114) | (0.0114) | (0.0114) | | | | | | | |

Table 4: Parameter estimates, log-likelihood, AIC, AICC, BIC, $W^*$ and $A^*$

| | Estimates | | | | Statistics | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | $\beta$ | $\delta$ | c | $\lambda$ | $-2\log L$ | AIC | AICC | BIC | W* | A* | SS |
| MOLLoGETE | 0.7611 | 60.8080 | 1.7027 | 0.0918 | 635.1020 | 643.1019 | 643.5230 | 653.5226 | 0.0249 | 0.1638 | 0.0225 |
| | (9.0487) | (29.0136) | (0.3377) | (1.1433) | | | | | | | |
| MOLLoGETE($\beta$,$\delta$,1,$\lambda$) | 0.0674 | 4.5284 | 1 | 0.8747 | 702.9472 | 709.0511 | 709.3011 | 716.8666 | 0.0261 | 0.1664 | 4.4743 |
| | (0.0364) | (23.0954) | ( 0.3188) | - | | | | | | | |
| MOLLoGETE(1,$\delta$,$c$,$\lambda$) | 1 | 3.6397 | 0.0621 | 0.1635 | 692.6966 | 698.7672 | 699.0172 | 706.5827 | 0.1896 | 1.1949 | 2.6256 |
| | - | (0.5404) | (0.1743) | (0.0255) | | | | | | | |
| MOLLoGETE (0,$\delta$,1,$\lambda$ ) | 0.00001 | 7.6556 | 1 | 0.7550 | 709.8156 | 713.8156 | 713.9393 | 719.0260 | 0.0496 | 0.3711 | 1.4941 |
| | - | (1.1523) | - | (16.6714) | | | | | | | |
| LLoG | - | - | 0.7346 | - | 855.0050 | 857.0050 | 857.0458 | 859.6102 | 0.0934 | 0.6763 | 13.22713 |
| | - | - | (0.0578) | - | | | | | | | |
| | $\alpha$ | $\beta$ | $\delta$ | $\lambda$ | | | | | | | |
| GD | 0.1843 | 8.1741 | 1.7438 | 22.9712 | 639.7148 | 647.7148 | 648.1358 | 658.1354 | 0.0584 | 0.4225 | 0.0486 |
| | (0.0205) | (0.0780) | (0.0542) | (0.0736) | | | | | | | |
| MOGETE | 4.1156 | 0.8399 | | 0.2601 | 641.424 | 647.4241 | 647.6741 | 655.2396 | 0.1072 | 0.6556 | 0.0734 |
| | (1.3538) | (8.3598) | | (2.9561) | | | | | | | |