Electronic Journal of Applied Statistical Analysis
EJASA, Electron. J. App. Stat. Anal.
http://siba-ese.unisalento.it/index.php/ejasa/index
e-ISSN: 2070-5948
DOI: 10.1285/i20705948v14n2p298

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Published: 20 November 2021

Electronic Journal of Applied Statistical Analysis
Vol. 14, Issue 02, November 2021, 298-304
DOI: 10.1285/i20705948v14n2p298

A Reward Problem to the Consecutive Heads In a Run

Hideo Hirose*

Biostatistics Center, Kurume University
Kurume, Fukuoka, Japan

Published: 20 November 2021

How many consecutive heads can we observe in a run of coin tossing of length n? Although the problem seems to be easy to answer, this would be actually a little bit tough when we try to find the solution straightforwardly. The expected number of consecutive heads in a run is $\frac{3n-2}{8}$ $(n \geq 2)$ using a recursive formula. However, if we define a solitary head coin such that a head coin is isolated by neighboring tail coin(s) in a run, the problem of how many solitary heads in a run can be solved easily. The expected number of solitary heads in a run is $\frac{n+2}{8}$ $(n \geq 2)$. Since the problem of solitary head coin becomes a dual problem of the above, the consequence of the problem of the consecutive heads is derived easily by considering the probability of a solitary coin appearance. Using this duality, we can solve much more complex problem such that how much the reward is expected in a run of coin tossing of length n if the reward is 2^{k-1} $(k \geq 2)$ when k consecutive heads appears. The expected reward is $\frac{1}{16}(n^2+3n-2)$, $(n \geq 2)$. Applying this result to adaptive e-learning systems, we can design the reward to promote self-study for students.

keywords: coin tossing, run, consecutive heads, solitary head coin, reward, dual problem.

*Corresponding author: hirose_hideo@kurume-u.ac.jp

ISSN: 2070-5948
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1 Introduction

There have been number of good discussions in coin tossing. Feller (1968) is most referred to, and very intriguing subjects are also discussed; e.g., Mood (1940), Bloom (1996), Finch (2003), Havil (2003), Gordon et al. (1986), Philippou and Marki (1986), Schilling (1990), Schuster (1994), Spencer (1986), are among them. Some are in ideally fair coins, and others are in actual coin, e.g., Keller (1986), Ford (1983). We deal with a problem of the number of consecutive heads in a run in this paper. However, in addition to this, we consider the reward problem to the consecutive heads.

Imagine that we are tossing a coin n times. The first question is how many consecutive heads we can observe in a run of coin tossing of length n? When $n = 3$, for example, the head (H) and tail (T) patterns are, HHH, HHT, HTH, ... , TTT; the number of all possible patterns is $2^3 = 8$. Among these eight patters, we find three consecutive heads patterns; two consecutive heads cases are HHT and THH, and three consecutive heads case is HHH. The number of consecutive heads is counted to be two, two or three to each case. When $n = 5$, there are no consecutive heads in a run of THTTH or HTHTH. When a run is HHTHH, we count the number of consecutive heads as $2 \times 2 = 4$; we observed two heads two times. When a run is THHHT, it is $3 \times 1 = 3$; we observed three heads one time. When THHHH, it is $4 \times 1 = 4$; we observed four heads one time. The number of consecutive heads is counted to be four, three or four to each case.

Next, we consider the rewards to the consecutive heads. In the above example cases, we define that the reward is two to each pattern of HHT or THH, four to the pattern of THHHT, and eight to the pattern of THHHH. That is, the reward soars exponentially as the number of consecutive heads becomes large. Here, the reward is calculated to be 2^{k-1} for k consecutive heads observation, where $k \geq 2$. The second question is how much reward we can expect in a run of coin tossing of length n. In the cases of $n = 2$ and $n = 3$, the expected rewards are $\frac{2}{2^2} = \frac{1}{2}$ and $\frac{2+2+4}{2^3} = 1$, respectively.

2 Expected Number of Consecutive Heads in A Run

To count the number of consecutive heads, we append one point to a head coin of consecutive heads and zero point otherwise. We first consider the case of $n = 3$ to grasp the problem. When $n = 3$, we get three point from HHH and two point each from HHT or THH, as shown in Figure 1; thus, the expected point in a run is $\frac{3+2+2}{2^3} = \frac{7}{8}$.

We define a_i such that

$$\begin{aligned} a_i = 1 &\qquad i\text{th flipped coin is head,} \\ a_i = 0 &\qquad i\text{th flipped coin is tail,} \end{aligned} \tag{1}$$

and b_i such that

$$\begin{aligned} b_i = 1 &\qquad i\text{th flipped coin is one of the consecutive heads,} \\ b_i = 0 &\qquad i\text{th flipped coin is a solitary head or tail.} \end{aligned} \tag{2}$$

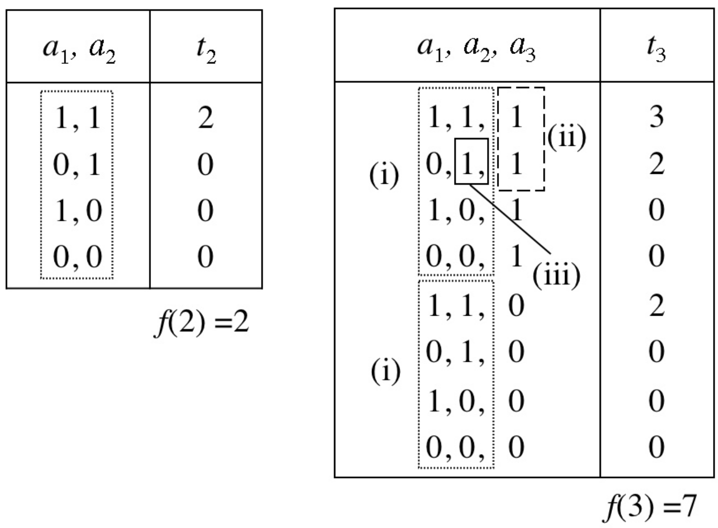

Figure 1: An example of a_i, t_i, $f(n)$, when $n = 2, 3$.

We, then, can define the point $t_n(a_1, a_2, ..., a_n)$ in a run by

$$t_n(a_1, a_2, ..., a_n) = \sum_{i=1}^{n} b_i. \tag{3}$$

By summing up t_n for all possible runs, we define the total possible point of

$$f(n) = \sum_{a_1, a_2, ..., a_n} t_n(a_1, a_2, ..., a_n). \tag{4}$$

As an example, we show a_i, t_n, and $f(n)$ for $n = 2, 3$ in Figure 1. Once we can obtain $f(n)$, the expected point, $E[t_n]$, becomes

$$E[t_n] = \frac{f(n)}{2^n}. \tag{5}$$

To obtain $f(n)$, we consider a recursive formula. The total point consists of the following three in n coin tossing:

1) Whatever the value of a_n is, $f(n-1)$ by $a_1, a_2, \ldots, a_{n-1}$ is taken into account of; i.e., $2f(n-1)$ is counted in $f(n)$ for $a_n = 0$ and $a_n = 1$. This is shown in (i) on the right in Figure 1.

2) If $a_{n-1} = 1$, and $a_n = 1$, then $b_n = 1$. Thus, 2^{n-2} point are counted in $f(n)$, since we have 2^{n-2} possible cases for $a_1, a_2, \ldots, a_{n-2}$. This is shown in (ii) on the right in Figure 1.

3) If $a_{n-2} = 0$, $a_{n-1} = 1$, and $a_n = 1$, then $b_{n-1} = 1$. When we deal with $n-1$ coin tossing, $b_{n-1} = 0$ if $a_{n-2} = 0$, $a_{n-1} = 1$. Thus, b_{n-1} changes its value from 0 to 1.

According to this, 2^{n-3} point are counted in $f(n)$ because we have 2^{n-3} possible cases for $a_1, a_2, \ldots, a_{n-3}$. This is shown in (iii) on the right in Figure 1.

Therefore, we have the recursive formula such that

$$f(n) = 2f(n-1) + 2^{n-2} + 2^{n-3}, \quad (n \geq 3). \tag{6}$$

This formula can be solved as follows:

$$\begin{aligned}
f(n) &= 2f(n-1) + 3 \cdot 2^{n-3} \\
&= 2(2f(n-2) + 3 \cdot 2^{n-4}) + 3 \cdot 2^{n-3} \\
&= 2^2 f(n-2) + 3 \cdot 2^{n-3} + 3 \cdot 2^{n-3} \\
&= 2^2 f(n-2) + 3 \cdot 2 \cdot 2^{n-3} \\
&= 2^2(2f(n-3) + 3 \cdot 2^{n-5}) + 3 \cdot 2 \cdot 2^{n-3} \\
&= 2^3 f(n-3) + 3 \cdot 2^{n-3} + 3 \cdot 2 \cdot 2^{n-3} \\
&= 2^3 f(n-3) + 3 \cdot 3 \cdot 2^{n-3} \\
&\vdots \\
&= 2^{n-4}(2f(3) + 3 \cdot 2^1) + 3(n-4)2^{n-3} \\
&= 2^{n-3} f(3) + 3(n-3)2^{n-3} \\
&= 2^{n-3}(f(3) + 3(n-3)) \\
&= 2^{n-3}(3n-2), \quad (n \geq 3).
\end{aligned} \tag{7}$$

Therefore, the expected point, $E[t_n]$, which is equivalent to the expected number of consecutive heads, C_n, becomes

$$E[t_n] = \frac{2^{n-3}(3n-2)}{2^n} = \frac{3n-2}{8} = C_n, \quad (n \geq 2), \tag{8}$$

in a run, because this formula also holds when $n = 2$.

Next, we want to consider the reward problem. Even though we look at Figure 1 very carefully, we cannot easily imagine how to solve this problem by constructing a similar recursive formula. However, the problem becomes easy if we change the idea to solve this problem. That is, we consider a dual problem shown below.

3 Probability That A Coin is A Solitary Head Coin

When $a_1 = 1$ and $a_2 = 0$, then the very first flipped coin is the solitary head coin. Whatever values the other a_i have, the probability that the first flipped coin is a solitary head coin is $\frac{1}{4}$ because the probability that $a_1 = 1$, $a_2 = 0$, and $a_i = 0, 1$ $(3 \leq i \leq n)$ is $\frac{2^{n-2}}{2^n}$. Let's consider here that we append point one only to the solitary head coin. Then, the expected point from this coin is $1 \times \frac{1}{4} = \frac{1}{4}$. This is also true for the very last flipped coin.

For the second flipped coin, it becomes a solitary head coin if $a_1 = 0$, $a_2 = 1$, and $a_3 = 0$, whatever values the other a_i have, where $4 \leq i \leq n$. Then, the expected point from this coin is $1 \times \frac{1}{8} = \frac{1}{8}$, and this is also true for $a_3, \ldots, a_{n-1}$.

Therefore, the total expected point for the solitary heads, which is equivalent to the expected number of solitary heads, S_n, in a run becomes,

$$S_n = 2 \times \frac{1}{4} + (n-2) \times \frac{1}{8} = \frac{n+2}{8}. \tag{9}$$

Considering that the problem of how many solitary heads we observe in a run becomes a dual problem for the original consecutive heads observation problem, the expected number of consecutive heads in a run, C_n, is

$$C_n = \frac{n}{2} - \frac{n+2}{8} = \frac{3n-2}{8}, \quad (n \geq 2). \tag{10}$$

4 Reward to Consecutive Heads in A Run

Next, we consider the case that we add reward to the appearance of the consecutive heads in a run. If k consecutive heads appears, we define that the reward attached to k consecutive heads becomes 2^{k-1}. We follow a precedent idea of solitary head coin.

We express the reward by $g(n)$. To obtain $g(n)$, we define sub-reward $g_k(n)$ to k consecutive heads; $g(n) = \sum_{k=2}^{n} g_k(n)$. For example, in the case of the two consecutive heads, $g_2(n)$ becomes,

$$g_2(n) = \{2 \times \frac{1}{2^{2+1}} + (n-3) \times \frac{1}{2^{2+2}}\} \times 2^{2-1} = \frac{n+1}{8}. \tag{11}$$

Similarly, $g_k(n)$ is expressed by,

$$\begin{aligned} g_k(n) &= \{2 \times \frac{1}{2^{k+1}} + (n-k-1) \times \frac{1}{2^{k+2}}\} \times 2^{k-1} \\ &= \frac{n-k+3}{8}, \ (2 \leq k \leq n-1), \end{aligned} \tag{12}$$

and

$$g_n(n) = (\frac{1}{2^n}) \times 2^{n-1} = \frac{1}{2}. \tag{13}$$

Thus,

$$g(n) = \sum_{k=2}^{n} g_k(n) = \frac{1}{16}(n^2 + 3n - 2), \ (n \geq 2). \tag{14}$$

Since the reward to k consecutive heads grows explosively as k increases, we are inclined to imagine that the expected rewards is also explosive. However, this is not true. The growth curve for the expected rewards follows only a quadratic function in coin tossing.

5 Application

A web-based learning system, e.g., Hirose (2016) provides a series of questions adaptively. That is, if a student solved one question, then the system selects the next question automatically such that its problem difficulty is best fit to the estimated ability of the student after estimating the ability using Bayes theory. The best fit means that the system selects a question from the question item bank such that the amount of information becomes to be maximum in the estimating procedure. Then, a student will solve the next question successfully with 50% probability.

To promote self-study using such e-learning systems, the system can provide a reward to a student. In such a situation, the result of the previous section can be directly applied such that a student can solve each question successfully with 50% probability and the reward is given in the same manner shown there. The expected reward curve is quadratic with respect to the number of questions.

However, if a student can solve a question with probability $p > \frac{1}{2}$, then the expected reward curve shows a different aspect. Equations $g_k(n,p)$, $g_n(n,p)$ and $g(n,p)$ similarly defined to $g_k(n)$, $g_n(n)$ and $g(n)$ become

$$\begin{aligned} g_k(n,p) &= \{2p^k(1-p) + (n-k-1)p^k(1-p)^2\} \times 2^{k-1}, \\ &\quad (2 \le k \le n-1), \qquad (15) \\ g_n(n,p) &= p^n \times 2^{n-1}, \qquad (16) \\ g(n,p) &= \sum_{k=2}^{n} g_k(n,p) \\ &= \frac{p^2}{2(1-2p)^2}\{(2p)^n + 4(n-1)(p-1)^2(1-2p) \\ &\quad +8p^2(p-1)\}, \ (n \ge 2). \qquad (17) \end{aligned}$$

In this situation, $g(n,p)$ will blow up as n becomes large unlike the case of $p = \frac{1}{2}$. This feature may be able to motivate students.

6 Concluding Remarks

We considered the problem of consecutive heads and its reward problem. The former is how many consecutive heads we can observe in a run of coin tossing of length n. The latter is how much reward we can expect in a run of coin tossing of length n. Although the problem of consecutive heads in a run can be solved using a recursive formula, its reward problem cannot be solved easily in a similar manner. However, the reward problem becomes easy if we change the idea to solve these problems. We considered a dual problem to the consecutive heads, i.e., the solitary head. Then, the solution can be easily obtained using the duality. Applying this result to adaptive e-learning systems, we can expect the amount of reward.

Acknowledgement

The author thanks to Drs. T. Noma and S. Kajihara for discussing this problem.

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