Electronic Journal of Applied Statistical Analysis
EJASA, Electron. J. App. Stat. Anal.
http://siba-ese.unisalento.it/index.php/ejasa/index
e-ISSN: 2070-5948
DOI: 10.1285/i20705948v12n1p176

Estimating Morgenstern type bivariate association parameter using a modified maximum likelihood method

By Al Kadiri, Migdadi

Published: 26 April 2019

Electronic Journal of Applied Statistical Analysis
Vol. 12, Issue 01, April 2019, 176-189
DOI: 10.1285/i20705948v12n1p176

Estimating Morgenstern type bivariate association parameter using a modified maximum likelihood method

Al Kadiri M.A.*[a] and Migdadi M. K.[b]

[a] *Department of Statistics, Yarmouk University, Irbid-Jordan.*
[b] *Common First Year Department, King Saud University, Riyadh-Saudi Arabia*

Published: 26 April 2019

This paper investigates estimating the association parameter of Morgenstern type bivariate distribution using a modified maximum likelihood method where the regular maximum likelihood methods failed to achieve estimation. The simple random sampling, concomitant of ordered statistics and bivariate ranked set sampling methods are used and compared. Efficiency and bias of the produced estimators are compared for two specific examples, Morgenstern type bivariate uniform and exponential distributions.

keywords: Morgenstern type bivariate distribution, Modified maximum likelihood estimation, Concomitant of ordered statistics, Bivariate ranked set sampling, efficiency.

1 Introduction

Suppose X and Y are two random variables with Cumulative Distribution Functions (CDF) $F_X(x)$ and $F_Y(y)$ respectively. Considerably, one can propose a few ways to merge these CDF's in a way that can accommodate the underlying relation of the data appropriately. Morgenstern (1956) suggested a general class of bivariate distribution function to join the CDF's that have a moderate association by introducing an association parameter α as follows:

$$F_{X,Y}(x, y, \alpha) = F_X(x)\ F_Y(y)\ \left[1 + \alpha\ (1 - F_X(x))\ (1 - F_Y(y))\right], \tag{1}$$

*Corresponding author: alkadiri-m@yu.edu.jo

ISSN: 2070-5948
http://siba-ese.unisalento.it/index.php/ejasa/index

where $-1 \leq \alpha \leq 1$. Thus, the corresponding probability density function (*pdf*) has the form:

$$f_{X,Y}(x,y,\alpha) = f_X(x) f_Y(y)\ [1 + \alpha\ (1 - 2F_X(x))\ (1 - 2F_Y(y))] \quad (2)$$

where $f_X(x)$ and $f_Y(y)$ are the marginal *pdf*'s for the variable X and Y respectively.

An example of this family, when proposing standard uniform marginals, is Morgenstern type bivariate uniform distribution. The *pdf* for this case can be written as:

$$f_{X,Y}(x,y,\alpha) = 1 + \alpha\ (1 - 2x))\ (1 - 2y)\ ;\ 0 < x < 1\,,\ 0 < y < 1. \quad (3)$$

Another member of this family is Morgenstern type bivariate exponential distribution that has the *pdf* as

$$f_{X,Y}(x,y,\alpha) = e^{-(x+y)}\ \left[1 + \alpha\ \left(1 - 2e^{-x}\right)\ \left(1 - 2e^{-y}\right)\right]\ ;\ 0 < x\,,\ 0 < y. \quad (4)$$

Here, the standard exponential marginals are assumed for random variables (X, Y).

A considerable amount of research focused in estimating the association parameter α of the Morgenstern type bivariate distribution under different sampling methods. Scaria and Nair (1999) used concomitant of order statistics approach to estimate the association parameter while Chacko and Thomas (2008) used ranked set sampling (RSS) method to estimate this parameter in Morgenstern type bivariate exponential density. Domma and Giardano (2015) estimated the association parameter for concomitant of m–generalized order statistics from generalized Farlie–Gumbel–Morgenstern distribution family. Tahmasebi and Jafari (2015) summarized the literature of estimating the association parameter of Morgenstern type bivariate gamma distribution using RSS.
In the literature, the Maximum Likelihood Estimate (MLE) method is not applicable for this type of distribution because the solution cannot be found. However, Genest et al. (1995) proposed Pseudo Maximum Likelihood estimation (PML) method to estimate the association (dependence) parameter in a similar type of distribution family, called copula type bivariate distributions.
The MLE method depends mainly on maximizing the likelihood of the joint distribution of the bivariate random variables w. r. t. the targeted parameter. Let(X_i, Y_i); $i = 1, ..., n$; be a Simple Random Sample (SRS) of size n, with joint Morgenstern density $f_{X_i,Y_i}(x, y)$ in (2) then, the likelihood function can be written as:

$$\begin{aligned} L(\alpha) &= \prod_{i=1}^{n} f_{X_i,Y_i}(x,y,\alpha) \\ &= \prod_{i=1}^{n} f_{X_i}(x) f_{Y_i}(y)[1 + \alpha(1 - 2F_{X_i}(x))(1 - 2F_{Y_i}(y))]. \end{aligned} \quad (5)$$

The MLE can be gained by maximizing the likelihood function in (5) or equivalently, maximizing its log function. Therefore, MLE estimator is equivalent to solving the likelihood equation:

$$\frac{\partial}{\partial \alpha} \log\ L(\alpha) = \sum_{i=1}^{n} \frac{\partial}{\partial \alpha} \log\left[f_{X_i}(x) f_{Y_i}(y)[1 + \alpha(1 - 2F_{X_i}(x))(1 - 2F_{Y_i}(y))]\right] = 0.$$

Simply, one can note that solving this likelihood equation is not applicable. So this paper proposes a few procedures to achieve α estimation via developing maximum likelihood approaches whereas introducing a set of sampling methods to the likelihood. Substantially, this paper estimates the association parameter of Morgenstern type bivariate distribution using PML estimation method where concomitant of order statistics sampling method is used. Also, it proposes a modified ML estimation method when a sampling method, known in the literature by Bivariate Ranked Sampling (BVRSS) technique, is used.

The BVRSS technique was firstly proposed by Al-Saleh and Zheng (2002) as a bivariate version of RSS. It can be used when we deal with two characteristics jointly. To describe this method, assume $(X, Y)$is a bivariate random pair with the joint probability density function $f_{X,Y}(x, y)$. We can summarize this method by the following steps.

Step 1: Select a random sample of size m^4 from the targeted population and divide it randomly into m^2 pools, each pool includes $m \times m$matrix of units. The units in the first pool are denoted by $\{(X_{ij}^{(1)}, Y_{ij}^{1}),\ i = 1, ..., m,\ j = 1, ..., m\}$, where $X_{ij}^{(1)}$ and $Y_{ij}^{(1)}$ are the j^{th} element of the i^{th} row in the first pool for the first characteristic and the second characteristic, respectively.

Step 2: For each row in the *first pool*, identify the minimum value based on first characteristic by judgement. In symbols $\{(X_{i(j)}^{(1)}, Y_{i[j]}^{(1)}),\ i = 1, ..., m,\ j = 1, ..., m\}$. Here, $Y_{i[j]}$is called the j^{th} concomitant random variable of the j^{th} ordered statistic$X_{i(j)}$.

Step 3: For the m minima obtained in Step 2, choose the pair that corresponds to the minimum value of the second characteristic, identified by judgment, for actual quantification. This step produces the first pair of BVRSS sample, label it by (1,1).

Step 4: Repeat Steps 2 and 3 for the *second pool*, but in Step 3, the pair that corresponds to the second minimum value with respect to the second characteristic is chosen for actual quantification. This produces the second pair of BVRSS sample, label it by (1,2).

Step 5: The process continues until the label $(m, m)$is yielded from the m^{th} *(last) pool.* In symbols $\{(X_{[i](j)}^{(k)}, Y_{(i)[j]}^{(k)}),\ i = 1, ..., m,\ j = 1, ..., m\}$ and $k = (j - 1)m + i$.

Step 6: If a larger sample size is needed, repeat the above steps r times (copies) to obtain a sample of size $n = rm^2$.

The above described method produces a BVRSS of size m^2, which are denoted by $\{(X_{[i](j)}^{(k)}, Y_{(i)[j]}^{(k)}),\ i = 1, ..., m,\ j = 1, ..., m$ and $k = (j - 1)m + i\}$, are independent but not identically distributed. Note that m^4 units are selected to quantify m^2 for actual measurement. Also, the m^4 sample units contribute the information with m^2 quantify units.

We draw attention in this paper, and as seen in equation (1) and (2), to note that all model parameters other than the association parameter are assumed to be known. Also, we propose two distribution examples that are Morgenstern type bivariate uniform and exponential distribution however; extending our method to a general internment is straightforward.

The following sections of this paper discuss the following issues. In Section 2, we estimate the association parameter using PMLE method when concomitant random variables are introduced in the likelihood model. In Section 3, we develop the likelihood function by introducing BVRSS sampling units to produce estimators. In Section 4, we illustrate our findings on some artificial examples as well as give a brief discussion. In the last section, Section 5, we give important conclusions.

2 Estimating α using ML method when concomitant of ordered statistics is used

For $1 \leq i \leq m$, suppose $(X_{(i)k}, Y_{[i]k})$be the i^{th}ordered statistics of the variable X and its corresponds concomitant of the variable Y. In some stages of this paper, we need to compare this sample with SRS sample of size n. So, it can be repeated in r copies such that $k = 1, \ldots, r$to attain sample size $n = r\ m$.
The joint pdf of $(X_{(i)}, Y_{[i]})$can be written as:

$$f_{X_{(i)},Y_{[i]}}(x,y,\alpha) = m \begin{pmatrix} m-1 \\ i-1 \end{pmatrix} f_{X_i,Y_i}(x,y,\alpha)\ \left[F_X(x)\right]^{i-1}\ \left[1-F_X(x)\right]^{m-i}. \quad (6)$$

Then, the log likelihood can be expressed as:

$$\begin{aligned} \ell_{FS}(\alpha) \quad = \quad & \sum_{k=1}^{r}\sum_{i=1}^{m} \log\Big(C_{m,i} f_{X_{ik}}(x) f_{Y_{ik}}(y)[1+\alpha(1-2F_{X_{ik}}(x))(1-2F_{Y_{ik}}(y))] \\ & [F_{X_{ik}}(x)]^{i-1}[1-F_{X_{ik}}(x)]^{m-i}\Big) \end{aligned} \quad (7)$$

where $C_{m,i} = m \begin{pmatrix} m-1 \\ i-1 \end{pmatrix} = \frac{m!}{(m-i)!(i-1)!}$.
Under specific regularity conditions, that were discussed by Abo-Eleneen and Nagaraja (2002), the MLE of αmaximizes the likelihood equation in (7) such that:

$$\frac{\partial}{\partial\alpha}\ell_{FS}(\alpha) = \sum_{k=1}^{r}\sum_{i=1}^{m} \frac{(1-2F_{X_{ik}}(x))(1-2F_{Y_{ik}}(y))}{[1+\alpha(1-2F_{X_{ik}}(x))(1-2F_{Y_{ik}}(y))]} = 0$$

which is identical to likelihood equation under the simple random sampling situation. This is because that:$\frac{\partial}{\partial\alpha}\log f_{X_{(i)},Y_{[i]}}(x,y,\alpha) = \frac{\partial}{\partial\alpha}\log f_{X_i,Y_i}(x,y,\alpha)$. This result can be written in general as:

$$\sum_{i=1}^{n} \frac{\frac{\partial}{\partial\alpha} f_{X,Y}(x,y,\alpha)}{f_{X,Y}(x,y,\alpha)} = \sum_{k=1}^{r}\sum_{i=1}^{m} \frac{\frac{\partial}{\partial\alpha} f_{X_{(i)k},Y_{[i]k}}(x,y,\alpha)}{f_{X_{(i)k},Y_{[i]k}}(x,y,\alpha)} = 0 \quad (8)$$

such that $n = rm$.
Pseudo Maximum Likelihood (PML) estimation method was established by Gong and Samaniego (1981). We describe their method using Morgenstern type bivariate distribution context. Under specific regularity conditions, the PML estimator of α, say $\hat{\alpha}_{PMLE}$, satisfies maxima of the likelihood equation by solving:

$$\frac{1}{rm}\frac{\partial}{\partial\alpha}\ell_{FS}(\alpha)=\frac{1}{rm}\sum_{k=1}^{r}\sum_{i=1}^{m}\frac{\partial}{\partial\alpha}\log f_{X_{ik},Y_{ik}}(x,y,\alpha)=0. \tag{9}$$

By expanding (9) using the first two terms of Taylor series, we get:

$$\left.\frac{1}{rm}\frac{\partial}{\partial\alpha}\ell_{FS}(\alpha)\right|_{\alpha=\hat{\alpha}_{PMLE}} \cong \frac{1}{rm}\sum_{k=1}^{r}\sum_{i=1}^{m}\frac{\partial}{\partial\alpha}\log f_{X_{ik},Y_{ik}}(x,y,\alpha) + (\hat{\alpha}_{PMLE}-\alpha)\frac{1}{rm}\sum_{k=1}^{r}\sum_{i=1}^{m}\frac{\partial^2}{\partial\alpha^2}\log f_{X_{ik},Y_{ik}}(x,y,\alpha)=0$$

Apply the last equation using Morgenstern type bivariate construction leads to:

$$\left.\frac{1}{rm}\frac{\partial}{\partial\alpha}\ell_{FS}(\alpha)\right|_{\alpha=\hat{\alpha}_{PMLE}} \cong \frac{1}{rm}\sum_{k=1}^{r}\sum_{i=1}^{m}\frac{(1-2F_{X_{ik}}(x))(1-2F_{Y_{ik}}(y))}{[1+\alpha(1-2F_{X_{ik}}(x))(1-2F_{Y_{ik}}(y))]} - (\hat{\alpha}_{PMLE}-\alpha)\frac{1}{rm}\sum_{k=1}^{r}\sum_{i=1}^{m}\frac{[(1-2F_{X_{ik}}(x))(1-2F_{Y_{ik}}(y))]^2}{[1+\alpha(1-2F_{X_{ik}}(x))(1-2F_{Y_{ik}}(y))]^2}=0.$$

As noted in the last likelihood equation, it depends on $F_{X_i}(x)$ and $F_{Y_i}(y)$. Genest et al. (1995) suggested a method in similar numerical situations by replacing the CDF's by their empirical CDF's multiplied by the coefficient $\frac{rm}{(rm+1)}$ such that:
$\bar{F}_Z(a)=\frac{rm}{(rm+1)}\sum_{i=1}^{s}I(z_i\leq a)$, where $I(.)$ is the indicator function on the random sample $z_1,...,z_s$ that has CDF $F_Z(.)$.
Genest et al. (1995) suggestion can be applied on estimating the association parameter for bivariate and multivariate families of distributions. Thus, we can replace $F_{X_i}(x)$ and $F_{Y_i}(y)$ in the last equation by their empirical CDF's $\bar{F}_X(x)$ and $\bar{F}_Y(y)$ respectively. This guides us to write the last likelihood equation in the form:

$$\left.\frac{1}{rm}\frac{\partial}{\partial\alpha}\ell_{FS}(\alpha)\right|_{\alpha=\hat{\alpha}_{PMLE}} \cong \frac{1}{rm}\sum_{k=1}^{r}\sum_{i=1}^{m}\frac{(1-2\bar{F}_{X_{ik}}(x))(1-2\bar{F}_{Y_{ik}}(y))}{[1+\alpha(1-2\bar{F}_{X_{ik}}(x))(1-2\bar{F}_{Y_{ik}}(y))]} -(\hat{\alpha}_{PMLE}-\alpha)\frac{1}{rm}\sum_{k=1}^{r}\sum_{i=1}^{m}\frac{[(1-2\bar{F}_{X_{ik}}(x))(1-2\bar{F}_{Y_{ik}}(y))]^2}{[1+\alpha(1-2\bar{F}_{X_{ik}}(x))(1-2\bar{F}_{Y_{ik}}(y))]^2}=0.$$

To simplify expression of this equation, we can write it as:

$$\sqrt{rm}(\hat{\alpha}_{PMLE}-\alpha)\equiv\sqrt{rm}\frac{W_{rm}(\alpha)}{U_{rm}(\alpha)}$$

where $W_{rm}(\alpha)=\frac{1}{rm}\sum_{k=1}^{r}\sum_{i=1}^{m}\frac{(1-2\bar{F}_{X_{ik}}(x))(1-2\bar{F}_{Y_{ik}}(y))}{[1+\alpha(1-2\bar{F}_{X_{ik}}(x))(1-2\bar{F}_{Y_{ik}}(y))]}$ and

$$U_{rm}(\alpha)=-\frac{1}{rm}\sum_{k=1}^{r}\sum_{i=1}^{m}\frac{[(1-2\bar{F}_{X_{ik}}(x))(1-2\bar{F}_{Y_{ik}}(y))]^2}{[1+\alpha(1-2\bar{F}_{X_{ik}}(x))(1-2\bar{F}_{Y_{ik}}(y))]^2}.$$

Consider $W_{rm}(\alpha)$ and $U_{rm}(\alpha)$ are two functions of the true value α. The following result is analogous to Genest et al. (1995).
Result 1: Under regularity conditions, $\sqrt{rm}(\hat{\alpha}_{PMLE} - \alpha)$ has asymptotic normal distribution with **zero** mean and Ω variance where $\Omega = \frac{var\left(\sqrt{rm}\, W_{rm}(\alpha)\right)}{U^2_{rm}(\alpha)}$.
Simply, we can note that $U_{rm}(\alpha)$ can be represented as:

$$U_{rm}(\alpha) = -\frac{1}{rm}\sum_{k=1}^{r}\sum_{i=1}^{m}\frac{\partial^2}{\partial\alpha^2}\log f_{X_{ik},Y_{ik}}(x, y, \alpha).$$

Asymptotically,

$$\begin{aligned} U_{rm}(\alpha) &\to -E\left[\frac{\partial^2}{\partial\alpha^2}\log f_{X_{ik},Y_{ik}}(x, y, \alpha)\right] \\ &= E\left[\frac{\partial}{\partial\alpha}\log f_{X,Y}(x, y, \alpha)\right]^2 \end{aligned}$$

which is the fisher information number, Stokes (1980).

3 Estimating α using modified maximum likelihood method using BVRSS units

Assume $\{(X_{[i](j)}, Y_{(i)[j]})\ ;\ i = 1, ..., m \text{ and } j = 1, ..., m\}$ be a BVRSS sample with size m^2 which originally came from Morgenstern type bivariate distribution in (2). The joint distribution of $(X_{[i](j)}, Y_{(i)[j]})$ was given by Al-Saleh and Zheng (2002) as:

$$\begin{aligned} f_{X_{[i](j)},Y_{(i)[j]}}(x, y) &= f_{Y_{(i)[j]}}(y)\frac{f_{X_{(j)}}(x) f_{Y|X}(y|x)}{f_{Y_{[j]}}(y)} \\ &= C_{m,i}[F_{Y_{[j]}}(y)]^{i-1}[1 - F_{Y_{[j]}}(y)]^{m-i} C_{m,j}[F_X(x)]^{j-1} \\ &\quad [1 - F_X(x)]^{m-j} f_{X,Y}(x, y, \alpha) \end{aligned} \tag{10}$$

where $F_{Y_{[j:m]}}(y)$ is the CDF of the j^{th} concomitant of Y. The coefficients $C_{m,i}$ and $C_{m,j}$ can be define similarly as in (7).
Introducing the bivariate Morgenstern environment to (10), we obtain:

$$\begin{aligned} f_{X_{[i](j)},Y_{(i)[j]}}(x, y) &= C_{m,i}\Big[F_{Y_{ij}}(y)[1 + \alpha\frac{m - 2j + 1}{m + 1}(1 - F_{Y_{ij}}(y))]\Big]^{i-1} \\ &\quad \Big[1 - F_{Y_{ij}}(y)[1 + \alpha\frac{m - 2j + 1}{m + 1}(1 - F_{Y_{ij}}(y))]\Big]^{m-i} \\ &\quad C_{m,j}[F_{X_{ij}}(x)]^{j-1}[1 - F_{X_{ij}}(x)]^{m-j} f_{X_{ij}}(x) \\ &\quad f_{Y_{ij}}(y)\Big[1 + \alpha(1 - 2F_{X_{ij}}(x))(1 - 2F_{Y_{ij}}(y))\Big] \end{aligned} \tag{11}$$

where the CD, $F_{Y_{[j]}}(y)= F_{Y_{ij}}(y)[1+\alpha\frac{m-2j+1}{m+1}(1-F_{Y_{ij}}(y))$ was derived by Scaria and Nair (1999).
In this section, we develop the ML estimation method to estimate the association parameter α using BVRSS samples $(X_{[i](j)k}, Y_{(i)[j]k})$ that have the likelihood in (11) such that $k=1,...,r$ where r is the replicates size. The sample size here is $n=rm^2$.
Accordingly, we can present the log likelihood function of (11) as:

$$\begin{aligned}\sum_{k=1}^{r}\sum_{j=1}^{m}\sum_{i=1}^{m}\log f_{X_{[i](j)k},Y_{(i)[j]k}}(x,y,\alpha) &= \sum_{k=1}^{r}\sum_{j=1}^{m}\sum_{i=1}^{m}\log f_{X_{(j)}}(x)+\sum_{k=1}^{r}\sum_{j=1}^{m}\sum_{i=1}^{m}\log\frac{f_{Y_{(i)[j]}}(y,\alpha)}{f_{Y_{[j]}}(y,\alpha)}\\ &+ \sum_{k=1}^{r}\sum_{j=1}^{m}\sum_{i=1}^{m}\log\frac{f_{X_i,Y_i}(x,y,\alpha)}{f_{X_i}(x)}.\end{aligned}$$

After eliminating the constant term ($i.e$ the first term) from the above log likelihood, we can fined the ML estimate of α by solving the following maximum likelihood equation:

$$\sum_{k=1}^{r}\sum_{j=1}^{m}\sum_{i=1}^{m}\frac{\frac{\partial}{\partial\alpha}f_{X_{[i](j)k},Y_{(i)[j]k}}(x,y,\alpha)}{f_{X_{[i](j)k},Y_{(i)[j]k}}(x,y,\alpha)}+\sum_{k=1}^{r}\sum_{j=1}^{m}\sum_{i=1}^{m}\frac{\frac{\partial}{\partial\alpha}\left[\frac{f_{Y_{(i)[j]k}}(y,\alpha)}{f_{Y_{[j]}}(y,\alpha)}\right]}{\frac{f_{Y_{(i)[j]k}}(y,\alpha)}{f_{Y_{[j]}}(y,\alpha)}}=0.$$

One more simplification step gives:

$$\sum_{k=1}^{r}\sum_{j=1}^{m}\sum_{i=1}^{m}\frac{\frac{\partial}{\partial\alpha}f_{X_{[i](j)k},Y_{(i)[j]k}}(x,y,\alpha)}{f_{X_{[i](j)k},Y_{(i)[j]k}}(x,y,\alpha)}+\sum_{k=1}^{r}\sum_{j=1}^{m}\sum_{i=1}^{m}\left[\frac{\frac{\partial}{\partial\alpha}f_{Y_{(i)[j]k}}(y,\alpha)}{f_{Y_{(i)[j]k}}(y,\alpha)}-\frac{\frac{\partial}{\partial\alpha}f_{Y_{[j]k}}(y,\alpha)}{f_{Y_{[j]k}}(y,\alpha)}\right]=0 \tag{12}$$

As can be noted, there is a difficulty in solving (12). So, Al-Saleh and Samawi (2005) suggested to apply a Modified ML (MML) estimation procedure. This procedure requires replacing the second term of (12) by its expectation.
Result 2: The expectation of the second term of (12) is zero and, thus, the modified likelihood equation becomes:

$$\sum_{k=1}^{r}\sum_{j=1}^{m}\sum_{i=1}^{m}\frac{\frac{\partial}{\partial\alpha}f_{X_{[i](j)k},Y_{(i)[j]k}}(x,y,\alpha)}{f_{X_{[i](j)k},Y_{(i)[j]k}}(x,y,\alpha)}=0. \tag{13}$$

Proof:
We take the expectation to the second term of equation (12) so, we get:

$$E\left(\sum_{k=1}^{r}\sum_{j=1}^{m}\sum_{i=1}^{m}\frac{\frac{\partial}{\partial\alpha}f_{Y_{(i)[j]k}}(y,\alpha)}{f_{Y_{(i)[j]k}}(y,\alpha)}\right)-E\left(\sum_{k=1}^{r}\sum_{j=1}^{m}\sum_{i=1}^{m}\frac{\frac{\partial}{\partial\alpha}f_{Y_{[j]k}}(y,\alpha)}{f_{Y_{[j]k}}(y,\alpha)}\right)$$

$$=\sum_{k=1}^{r}\sum_{j=1}^{m}\sum_{i=1}^{m}\int\frac{\frac{\partial}{\partial\alpha}f_{Y_{(i)[j]k}}(y,\alpha)}{f_{Y_{(i)[j]k}}(y,\alpha)}f_{Y_{(i)[j]k}}(y,\alpha)\,dy-\sum_{k=1}^{r}\sum_{j=1}^{m}\sum_{i=1}^{m}\int\frac{\frac{\partial}{\partial\alpha}f_{Y_{[j]k}}(y,\alpha)}{f_{Y_{[j]k}}(y,\alpha)}f_{Y_{[j]k}}(y,\alpha)\,dy.$$

Under the following regularity conditions:
$\frac{\partial}{\partial\alpha}\int f_{Y_{[j]}}(y,\alpha)\,dy = \int \frac{\partial}{\partial\alpha} f_{Y_{[j]}}(y,\alpha)\,dy$ and $\frac{\partial}{\partial\alpha}\int f_{Y_{(i)[j]k}}(y,\alpha)\,dy = \int \frac{\partial}{\partial\alpha} f_{Y_{(i)[j]k}}(y,\alpha)\,dy$
and by using the facts:
$\sum_{j=1}^{m}\sum_{i=1}^{m} f_{Y_{[j]}}(y,\alpha) = m^2 f_Y(y)$ and $\sum_{j=1}^{m}\sum_{i=1}^{m} f_{Y_{(i)[j]}}(y,\alpha) = m^2 f_Y(y)$
thus, the proof can be finished straightforwardly.
A general result which was concluded by Abo-Eleneen and Nagaraja (2002) that ML estimators of α using SRS and concomitant of ordered statistics have the same expression. This result is summarized in equation (8) above. Analogously, we conclude that it has the same expression when using the BVRSS which is equal to equation (13).
According to this result, we consider the following general procedure to estimate α. We solve (13) to find an MML estimator for α using BVRSS units, say $\hat{\alpha}_{MMLE,BVRSS}$. Then, we can use either SRS or concomitant of ordered statistics sampling units in the same solution to get ML estimators. Call them $\hat{\alpha}_{MLE,SRS}$ and $\hat{\alpha}_{MLE,COS}$ respectively.
Before concluding this section, we simplify presentation of the modified likelihood equation in (13). This simplification can make it easier when working with specific examples as seen in Section (4).
Firstly, the joint pdf can be represented as:

$$f_{X_{[i](j)},Y_{(i)[j]}}(x,y,\alpha) = T_{ij}(x,y)[W_{ij}(\alpha)]^{i-1}[1-W_{ij}(\alpha)]^{m-i}Z_{ij}(\alpha) \tag{14}$$

where $T_{ij}(x,y) = C_{m,i}C_{m,j}[F_{X_{ij}}(x)]^{j-1}[1-F_{X_{ij}}(x)]^{m-j} f_{X_{ij}}(x) f_{Y_{ij}}(y)$,
$W_{ij}(\alpha) = F_{Y_{ij}}(y)\left[1+\alpha\, A_{j,m}(1-F_{Y_{ij}}(y))\right]$, $Z_{ij}(\alpha) = 1+\alpha\left((1-2F_{X_{ij}}(x))(1-2F_{Y_{ij}}(y))\right)$
and $A_{j,m} = \frac{m-2j+1}{m+1}$. *Hint*: For general sampling setup, we expressed the BVRSS units by X_{ij} and Y_{ij} instead.
So, the modified likelihood equation can be written as:

$$\sum_{k=1}^{r}\sum_{j=1}^{m}\sum_{i=1}^{m} \frac{\frac{\partial}{\partial\alpha} f_{X_{[i](j)k},Y_{(i)[j]k}}(x,y,\alpha)}{f_{X_{[i](j)k},Y_{(i)[j]k}}(x,y,\alpha)} = \sum_{k=1}^{r}\sum_{j=1}^{m}\sum_{i=1}^{m}\left[(i-1)\frac{\frac{\partial}{\partial\alpha}W_{ij}(\alpha)}{W_{ij}(\alpha)}Z_{ij}(\alpha) + (m-i)\frac{\frac{\partial}{\partial\alpha}(1-W_{ij}(\alpha))}{(1-W_{ij}(\alpha))}Z_{ij}(\alpha) + \frac{\partial}{\partial\alpha}Z_{ij}(\alpha)\right] = 0. \tag{15}$$

4 Examples and simulation studies

In this section, we apply the general procedure that has been mentioned at the end of Section 3 to estimate the association parameter for Morgenstern type bivariate uniform and exponential distributions as specific examples. These distributions have been defined in (3) and (4), respectively.
We used the MML estimation method to estimate α. We solved the MML estimation equation in (13) by introduced one of the three sampling procedures, SRS, concomitant or BVRSS on this likelihood. We called the yielded estimators by: $\hat{\alpha}_{MLE,SRS}$, $\hat{\alpha}_{MLE,COS}$ and $\hat{\alpha}_{MMLE,BVRSS}$ respectively.

Without loss of generality, we set the nonparametric case for these two specific distributions and we considered all model parameters – other than the association parameter – as known values. Also, since the modified likelihood equation in (13) does not affect by number of copies (r) of BVRSS samples, we considered $r = 1$. However, in the concomitant of ordered statistics sampling, we proposed r as needed.

4.1 Numerical examples

Example 1: The Morgenstern type bivariate uniform density
Assume the random variables X and Y have originally standard uniform marginals. This produces the Morgenstern model in (3). To construct this model in the form as (14), one can conclude that
$W_{ij}(\alpha) = y_{ij}\,[1 + \alpha\, A_{j,m}(1 - y_{ij})]$ and $Z_{ij}(\alpha) = 1 + \alpha(1 - 2x_{ij})(1 - 2y_{ij})$.
Thus, the MML estimator of αsatisfies the maximum likelihood equation:

$$\sum_{k=1}^{m}\sum_{j=1}^{m}\sum_{i=1}^{m}\Bigg[(i-1)\qquad \frac{A_{j,m}y_{ij}(1-y_{ij})}{y_{ij}[1+\alpha A_{j,m}(1-y_{ij})]} - (m-i)\frac{A_{j,m}y_{ij}(1-y_{ij})}{1-y_{ij}[1+\alpha A_{j,m}(1-y_{ij})]} + \frac{(1-2x_{ij})(1-2y_{ij})}{1+\alpha A_{j,m}(1-2x_{ij})(1-2y_{ij})}\Bigg] = 0. \tag{16}$$

Example 2: The Morgenstern type bivariate exponential density
Assume the random variables X and Y have originally standard exponential marginals. This produces the Morgenstern model in (4). To construct this model in the form as (14), one can conclude that:
$W_{ij}(\alpha) = (1 - e^{-y_{ij}})\,[1 + \alpha\, A_{j,m}\, e^{-y_{ij}}]$ and $Z_{ij}(\alpha) = 1 + \alpha(1 - 2e^{-x_{ij}})(1 - 2e^{-y_{ij}})$.
Therefore, the MML estimator of α satisfies the maximum likelihood equation:

$$\sum_{k=1}^{m}\sum_{j=1}^{m}\sum_{i=1}^{m}\Bigg[(i-1)\qquad \frac{A_{j,m}e^{-y_{ij}}(1-e^{-y_{ij}})}{(1-e^{-y_{ij}})[1+\alpha A_{j,m}e^{-y_{ij}}]} - (m-i)\frac{-A_{j,m}e^{-y_{ij}}(1-e^{-y_{ij}})}{1-(1-e^{-y_{ij}})[1+\alpha A_{j,m}e^{-y_{ij}}]} + \frac{(1-2e^{-x_{ij}})(1-2e^{-y_{ij}})}{1+\alpha(1-2e^{-x_{ij}})(1-2e^{-y_{ij}})}\Bigg] = 0. \tag{17}$$

4.2 Simulation studies

To illustrate our findings in this paper, we demonstrated simulation studies over the two specific distributions above. Firstly, we generated SRS samples with size n. Then, we used these samples to maximize the likelihood equations in (16) and (17). This produced the estimators: $\hat{\alpha}_{MLE,SRS}$ for the proposed distributions. Then, we generated the concomitant of ordered statistics samples with size $n = r\ m$. Again, we employed these samples to maximize the likelihood equations in (16) and (17) to produce the estimators: $\hat{\alpha}_{MLE,COS}$ for the two distributions. Finally, we generated BVRSS samples with size

$n = rm = m^2$. These samples were implemented to maximize the likelihood equations in (16) and (17) to produce the estimators: $\hat{\alpha}_{MMLE,BVRSS}$ for the two distributions.

A general procedure to generate Morgenstern bivariate random samples from specific marginals was discussed by Migdadi (2018).

Different combinations of sample sizes as well as association parameter were assumed such that $m = 3$, 5, 7 and 9 while values of the association parameter were $\alpha = 0.25$, 0.50, 0.75 and 0.99. We used Mathematica software to solve equations in (16) and (17) after introducing the simulated samples to get parameter estimators. We computed efficiency and bias of these estimators. Generally, we can define the efficiency of the estimator $\widehat{\theta}_1$ compared to the estimator $\widehat{\theta}_2$ by: $eff\left(\widehat{\theta}_1, \widehat{\theta}_2\right) = Var(\widehat{\theta}_2)/Var(\widehat{\theta}_1)$. For the case of unbiased estimators, the variance becomes Mean Square Error (MSE). It can be noted that the efficiency is greater than 1 when $Var\left(\widehat{\theta}_1\right) < Var\left(\widehat{\theta}_2\right)$. This means that $\widehat{\theta}_1$ is more efficient than $\widehat{\theta}_2$.

A set of simulation runs were conducted to compute efficiency and bias of the three estimators, which are $\hat{\alpha}_{MLE,SRS}$ and $\hat{\alpha}_{MLE,COS}$ and $\hat{\alpha}_{MMLE,BVRSS}$, under the two specific distributions. The results are summarized in the following tables: Table 1, Table 2, Table 3 and Table 4.

Table 1: Efficiency and Bias of $\hat{\alpha}_{MLE,COS}$ compared with $\hat{\alpha}_{MLE,SRS}$ for Morgenstern bivariate uniform distribution.

		$\mid \alpha \mid$			
$\boldsymbol{m}$		0.25	0.50	0.75	0.99
3	*Efficiency*	1.0197	1.1332	1.2006	1.2826
	$\lvert Bias_{SRS}\rvert$	0.0643	0.0481	0.0208	0.0085
	$\lvert Bias_{COS}\rvert$	0.0674	0.0411	0.0210	0.0054
5	*Efficiency*	1.1172	1.2184	1.4079	1.4997
	$\lvert Bias_{SRS}\rvert$	0.0532	0.0218	0.0100	0.0006
	$\lvert Bias_{COS}\rvert$	0.0467	0.0355	0.0076	0.0002
7	*Efficiency*	1.3177	1.3999	1.4361	1.5194
	$\lvert Bias_{SRS}\rvert$	0.0172	0.0090	0.0092	0.0003
	$\lvert Bias_{COS}\rvert$	0.0085	0.0063	0.0014	0.0004
9	*Efficiency*	1.3764	1.4426	1.5093	1.5931
	$\lvert Bias_{SRS}\rvert$	0.0075	0.0048	0.0051	0.0002
	$\lvert Bias_{COS}\rvert$	0.0036	0.0007	0.0003	0.0002

Table 2: Efficiency and Bias of $\hat{\alpha}_{MMLE,BVRSS}$ compared with $\hat{\alpha}_{MLE,SRS}$ for Morgenstern bivariate uniform Distribution.

		$\mid \alpha \mid$			
$\boldsymbol{m}$		0.25	0.50	0.75	0.99
3	*Efficiency*	1.2866	1.3622	1.4733	1.5430
	$\lvert Bias_{SRS} \rvert$	0.0924	0.0617	0.0682	0.0541
	$\lvert Bias_{BVRSS} \rvert$	0.0731	0.0475	0.0156	0.0094
5	*Efficiency*	1.3122	1.3768	1.5312	1.6723
	$\lvert Bias_{SRS} \rvert$	0.0557	0.0228	0.0101	0.0073
	$\lvert Bias_{BVRSS} \rvert$	0.0426	0.0205	0.0114	0.0052
7	*Efficiency*	1.6597	1.7856	1.9775	2.0123
	$\lvert Bias_{SRS} \rvert$	0.0155	0.0072	0.0039	0.0074
	$\lvert Bias_{BVRSS} \rvert$	0.0129	0.0058	0.0021	0.0001
9	*Efficiency*	1.7434	1.8052	1.9913	2.2789
	$\lvert Bias_{SRS} \rvert$	0.0386	0.0410	0.0024	0.0013
	$\lvert Bias_{BVRSS} \rvert$	0.0163	0.0037	0.0008	0.0003

5 Conclusions

A main benefit of BVRSS is originally to achieve more information from ordered observations after quantification rather than SRS. In this research, we apply this approach to estimate parameters for Morgenstern type bivariate specific distributions.

As noted from tables in the simulation section, we can conclude that the produced estimators, $\hat{\alpha}_{MMLE,BVRSS}$ and $\hat{\alpha}_{MLE,COS}$, are more efficient than $\hat{\alpha}_{MLE,SRS}$ and under the two proposed distributions. Simply, we can realise that $\hat{\alpha}_{MMLE,BVRSS}$ is the most efficient than all other estimators. Also, it can be seen that efficiency of estimators increased as the sample size increased as well as the association grew up.

Importantly, since value of bias in all iterations was very tiny, one can note that MML estimation procedure we improved in this paper to estimate the association parameter is admirable.

Further investigation can potentially study the effect of working with a single characteristic instead of rank both. This issue practically raises in the case of moderate or small association between variables. Therefore, costs can be minimized. A suggested research that can be used to achieve this aim is, for example, Al-Saleh and Al Kadiri (2000).

Table 3: Efficiency and Bias of $\hat{\alpha}_{MLE,COS}$ compared with $\hat{\alpha}_{MLE,SRS}$ for Morgenstern bivariate exponential distribution.

		$\mid \alpha \mid$			
$\boldsymbol{m}$		0.25	0.50	0.75	0.99
3	*Efficiency*	1.0001	1.0674	1.1163	1.1792
	$\lvert Bias_{SRS} \rvert$	0.0841	0.0921	0.0374	0.0311
	$\lvert Bias_{COS} \rvert$	0.0886	0.0724	0.0331	0.0287
5	*Efficiency*	1.0951	1.1866	1.1994	1.2227
	$\lvert Bias_{SRS} \rvert$	0.0376	0.0199	0.0071	0.0047
	$\lvert Bias_{COS} \rvert$	0.0054	0.0075	0.0018	0.0008
7	*Efficiency*	1.1349	1.2017	1.2685	1.3004
	$\lvert Bias_{SRS} \rvert$	0.0079	0.0051	0.0055	0.0016
	$\lvert Bias_{COS} \rvert$	0.0052	0.0006	0.0005	0.0004
9	*Efficiency*	1.1764	1.2861	1.3184	1.3761
	$\lvert Bias_{SRS} \rvert$	0.0009	0.0006	0.0001	0.0002
	$\lvert Bias_{COS} \rvert$	0.0005	0.0006	0.0004	0.0003

Acknowledgement

The authors would like to thank the editor and the referees for their helpful suggestions and comments which improved this paper.

Table 4: Efficiency and Bias of $\hat{\alpha}_{MMLE,BVRSS}$ compared with $\hat{\alpha}_{MLE,SRS}$ for Morgenstern bivariate exponential distribution.

		$\mid \alpha \mid$			
$\boldsymbol{m}$		0.25	0.50	0.75	0.99
3	*Efficiency*	1.0759	1.2940	1.3339	1.4267
	$\lvert Bias_{SRS}\rvert$	0.0916	0.0499	0.0061	0.0058
	$\lvert Bias_{BVRSS}\rvert$	0.0653	0.0471	0.0037	0.0039
5	*Efficiency*	1.1268	1.3973	1.4329	1.5612
	$\lvert Bias_{SRS}\rvert$	0.0051	0.0037	0.0031	0.0017
	$\lvert Bias_{BVRSS}\rvert$	0.0032	0.0018	0.0011	0.0008
7	*Efficiency*	1.3776	1.4834	1.5105	1.6012
	$\lvert Bias_{SRS}\rvert$	0.0021	0.0016	0.0007	0.0008
	$\lvert Bias_{BVRSS}\rvert$	0.0018	0.0021	0.0001	0.0004
9	*Efficiency*	1.4689	1.5370	1.6228	1.9821
	$\lvert Bias_{SRS}\rvert$	0.0027	0.0015	0.0011	0.0005
	$\lvert Bias_{BVRSS}\rvert$	0.0012	0.0009	0.0003	0.0001

References

Abo-Eleneen, Z. and Nagaraja, H. (2002). Fisher information in an order statistic and its concomitant. *Ann. Inst. Statist. Math.*, 54(3):667–680.

Al-Saleh, M. and Al Kadiri, M. A. (2000). Double-ranked set sampling. *Statistics & Probability Letters*, 48:205–212.

Al-Saleh, M. and Samawi, H. (2005). Estimation of the correlation coefficient using bivariate ranked set sampling with application to the bivariate normal distribution. *Communications in Statistics-Theory and Methods*, 34:875–889.

Al-Saleh, M. and Zheng, G. (2002). Estimation of bivariate characteristics using ranked set sampling. *Australian and New Zealand Journal of Statistic*, 44:221–232.

Chacko, M. and Thomas, Y. (2008). Estimation of a parameter of morgenstern type bivariate exponential by ranked set sampling. *Annals of the Institute of Statistical Mathematics*, 60:273–300.

Domma, F. and Giardano, S. (2015). Concomitants of m-generalized order statistics from generalized farlie–gumbel–morgenstern distribution family. *Journal of Computational and Applied Mathematics*, 294:413–435.

Genest, C., K, G., and L, R. (1995). A semiparametric estimation procedure of dependence parameters in multivariate families of distributions. *Biometrika*, 82:543–552.

Gong, G. and Samaniego, F. (1981). Maximum likelihood estimation: theory and application. *Annals of Statistics*, 9:861–869.

Migdadi, M. (2018). On estimating morgenstern type bivariate distribution using bivariate ranked set sampling. *Master Thesis, Yarmouk Universit, Irbid, Jordan.*

Morgenstern, D. (1956). Simple examples of two-dimensional distributions. *Statistik*, 8:234–235.

Scaria, J. and Nair, N. (1999). On concomitants of order statistics from morgenstern family. *Biometrical Journal*, 41:483–489.

Stokes, S. (1980). Ranked set sampling with concomitant variables. *Communications in Statistics*, 6:1207–1211.

Tahmasebi, S. and Jafari, A. (2015). A review on unbiased estimators of a parameter from morgenstern type bivariate gamma distribution using ranked set sampling. *Azerbaijan Journal of Mathematics*, 5:3–12.