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Estimation of circular-circular probability distribution

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This paper aims to introduce an estimation algorithm for the joint density of a circular-circular random variable, which is expressible in the form as discussed by Fernández-Durán (2007). The performance of the algorithm has been checked with the help of a simulation study and it is found to perform efficiently even for small sample sizes. Furthermore, the performance of the proposed algorithm is compared with that of an existing method of density estimation and is found to perform better than the existing one, which is indicated by the higher mean square error values for the estimates obtained by the latter method. Finally, the application of the algorithm is displayed by estimating the joint density of a circular-circular random variable arising in a real-life data set.

keywords: Joint circular-circular density, Circular-circular random variable, Estimation algorithm, Simulation study.

1 Introduction

Often, there arise situations that demand the study of the joint relationship between two circular variables or the assessment of the effect one circular variable is imposing upon the other. For example, in environmental studies, a researcher may want to see if the wind direction during the morning affects that measured during the evening or if there is any association between the pattern of hourly arrivals of customers at two different shops (Fernández-Durán, 2007), which can be modeled as a circular random variable. The estimation of the joint density will enable us to measure the combined effect of the two variables.

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A circular random variable, say Θ is the one which assumes values in the unit circle and its density function $f(\theta)$ is such that $f(\theta) = f(\theta + p.2\pi)$, where p is an integer. Consequently, for a circular-circular random variable, the support is a unit torus or any subset of it (Mardia and Jupp, 2000). Wehrly and Johnson (1980) expressed the possibility of representing the joint density function of a pair of circular random variables, say (Θ_1, Θ_2) in the following form:

$$f_{\Theta_1,\Theta_2}(\theta_1, \theta_2) = 2\pi g\left[2\pi\left\{F_{\Theta_1}(\theta_1) + F_{\Theta_2}(\theta_2)\right\}\right] f_{\Theta_1}(\theta_1) f_{\Theta_2}(\theta_2) \tag{1}$$

$f_1(.)$ and $f_2(.)$ being the density functions, $F_{\Theta_1}(\theta_1)$ and $F_{\Theta_2}(\theta_2)$, the distribution functions of Θ_1 and Θ_2 and $g(.)$ being the joining circular density, which is also sometimes referred to as the binding density. This is a special case of the full Wehrly and Johnson model (Wehrly and Johnson, 1980), given by

$$f_{\Theta_1,\Theta_2}(\theta_1, \theta_2) = 2\pi g\left[2\pi\left\{F_{\Theta_1}(\theta_1) - qF_{\Theta_2}(\theta_2)\right\}\right] f_{\Theta_1}(\theta_1) f_{\Theta_2}(\theta_2) \tag{2}$$

where $q \in \{-1, 1\}$ is non-random. (1) is obtained from (2) by putting $q = -1$.

2 A brief review of the circular analogue of copula function

The Sklar's theorem (Sklar, 1959) enables us to write the joint density function $f(.,.)$ of two linear random variables in terms of the copula density $c(.,.)$ as

$$f(x, y) = c(F_1(x), F_2(y)) f_1(x) f_2(y) \qquad x, y \in R \tag{3}$$

Copula functions have found wide applications in finance and econometrics (Giorgia, 2015). In the context of circular statistics, a general bivariate circular density on the unit torus can be written in terms of the circular copula density, known as the circula density 'c' (Jones et al., 2015) and marginal densities and distribution functions f_1, f_2, F_1 and F_2 respectively as

$$f_{\Theta_1,\Theta_2}(\theta_1, \theta_2) = 4\pi^2 f_{\Theta_1}(\theta_1) f_{\Theta_2}(\theta_2) c(2\pi F_1(\theta_1), 2\pi F_2(\theta_2)) \tag{4}$$

Combining equations (1) and (4), we find that the circula density can be linked with the joining circular density by the relation

$$c(2\pi F_1(\theta_1), 2\pi F_2(\theta_2)) = \frac{1}{2\pi} g\left[2\pi F_1(\theta_1) + 2\pi F_2(\theta_2)\right] \tag{5}$$

If the binding density in (1) is uniform on the full circle, the joint density reduces to:

$$f_{\Theta_1,\Theta_2}(\theta_1, \theta_2) = f_{\Theta_1}(\theta_1) f_{\Theta_2}(\theta_2)$$

This implies that the marginal densities are independent. Also, the circula reduces to the independence circula whose density is:

$$c(2\pi F_1(\theta_1), 2\pi F_2(\theta_2)) = \tfrac{1}{4\pi^2}$$

Conversely, if Θ_1 and Θ_2 are independent, then we have

$$c\left(2\pi F_1\left(\theta_1\right), 2\pi F_2\left(\theta_2\right)\right) = \tfrac{1}{4\pi^2}$$

Thus, if Θ_1 and Θ_2 are independent, we get from (5),

$$\begin{aligned}
&\Rightarrow \frac{1}{2\pi} g\left[2\pi F_1\left(\theta_1\right) + 2\pi F_2\left(\theta_2\right)\right] = \frac{1}{4\pi^2} \\
&\Rightarrow g\left[2\pi F_1\left(\theta_1\right) + 2\pi F_2\left(\theta_2\right)\right] = \frac{1}{2\pi}
\end{aligned}$$

Thus, if the marginal densities are independent, the binding density is uniform on the full circle.

3 Estimation Algorithm

Equation (1) shows that for the joint density estimation, three densities require to be estimated beforehand viz. the marginal densities of Θ_1 and Θ_2 (and also, the corresponding distribution functions) and the joining circular density $g\left(.\right)$. Suppose we have a circular-circular sample consisting of the observations $\{\theta_{1i}, \theta_{2i}\}_{i=1}^{n}$. A simple and straight-forward procedure for estimating $f_{\Theta_1,\Theta_2}\left(\theta_1, \theta_2\right)$ is presented through the following general algorithm:

3.1 Proposed Estimation algorithm of the joint density of two circular random variables

- **Step 1.** Estimate the marginal densities $f_1\left(.\right)$ and $f_2\left(.\right)$ and the respective distribution functions $F_{\Theta_1}\left(\theta_1\right)$ and $F_{\Theta_2}\left(\theta_2\right)$ of Θ_1 and Θ_2.

- **Step 2.** Work out the joint sample of Θ_1 and Θ_2, $\left[2\pi\left\{\hat{F}_{\Theta_1}\left(\theta_{1i}\right) + \hat{F}_{\Theta_2}\left(\theta_{2i}\right)\right\}\right]_{i=1}^{n}$, followed by estimation of the joining circular density $g\left(.\right)$. Here, $\hat{F}_{\Theta_1}\left(.\right)$ and $\hat{F}_{\Theta_2}\left(.\right)$ are the estimated distribution functions of Θ_1 and Θ_2 obtained in step 1.

- **Step 3.** Obtain the joint circular-circular density estimator as

$$\hat{f}_{\Theta_1,\Theta_2}\left(\theta_1, \theta_2\right) = 2\pi\hat{g}\left[2\pi\left\{\hat{F}_{\Theta_1}\left(\theta_1\right) + \hat{F}_{\Theta_2}\left(\theta_2\right)\right\}\right]\hat{f}_{\Theta_1}\left(\theta_1\right)\hat{f}_{\Theta_2}\left(\theta_2\right) \tag{6}$$

  where $\hat{g}\left(.\right)$ is the estimator of the joining circular density obtained in step 2 and $\hat{f}_{\Theta_1}\left(\theta_1\right)$, $\hat{f}_{\Theta_2}\left(\theta_2\right)$ are the estimated marginal densities of Θ_1 and Θ_2 as obtained in step 1.

An advantage of the proposed method is that the joining circular density estimate $\hat{g}\left(.\right)$ will indicate if Θ_1 and Θ_2 are associated (or independent). If $g\left(.\right)$ is uniform, then Θ_1 and Θ_2 are independent and vice versa.

In this paper, the marginal densities $f_{\Theta_1}(.)$ and $f_{\Theta_2}(.)$ and the joining circular density $g(.)$ have been estimated by the Maximum Likelihood method, where the estimates are obtained by maximizing the log-likelihood function of the sample.
The algorithm also embodies the other methods of density estimation such as method of moments (where the estimates are obtained by equating the sample trigonometric moments to the population trigonometric moments), method of scoring (where the estimates are obtained by solving the maximum likelihood equations numerically). In case any of the existing circular distribution fails to fit the data, the non-parametric kernel density estimation method is taken resort to for estimation of the marginal densities.

3.2 Another (existing) method of circular-circular density estimation

An existing method of estimating the joint density in (1) which is due to Jones et al. (2015), consists in estimating the parameters of the joint distribution by maximizing the log-likelihood function with respect to variations in the parameters of the distribution, followed by substitution of the estimated parameters and the marginal density and distribution function forms in the density in (1). Therein, the mean direction parameter of the binding density is set at the value equal to 0. The log-likelihood function of a random sample $(\theta_{11},\theta_{21}),(\theta_{12},\theta_{22}),\ldots,(\theta_{1n},\theta_{2n})$ from the distribution with density (1) is given by

$$l(\tau)=n\log(2\pi)+\sum_{i=1}^{n}\log\left[f_{\Theta_1}(\theta_{1i})\right]+\sum_{i=1}^{n}\log\left[f_{\Theta_2}(\theta_{2i})\right]+\sum_{i=1}^{n}\log\left[g\left[2\pi\left\{F_{\Theta_1}(\theta_{1i})+F_{\Theta_2}(\theta_{2i})\right\}\right]\right]$$

where τ is the vector containing the parameters of the distribution.

4 Simulation study

The performance of the proposed estimation algorithm and the existing estimation method of joint circular-circular density will be evaluated using the following two examples and their performances will be compared.

- **Example 1:** Let $f_{\Theta_1}(.)$ be the von Mises density with parameters μ_1 and κ_1 and $f_{\Theta_2}(.)$ be the von Mises density with parameters μ_2 and κ_2. Let us take the joining density $g(.)$ as the von Mises density with parameters μ and κ. Then assuming that the joint density of Θ_1 and Θ_2 admits the representation in (1), the circular-circular density with marginals $f_{\Theta_1}(.)$ and $f_{\Theta_2}(.)$ is given by

$$\begin{aligned}f_{\Theta_1,\Theta_2}(\theta_1,\theta_2)=\frac{1}{2\pi I_0(\kappa)\,I_0(\kappa_1)\,I_0(\kappa_2)}&\exp\{\kappa\cos[2\pi F_{\Theta_1}(\theta_1)+\\&2\pi F_{\Theta_2}(\theta_2)-\mu]+\kappa_1\cos(\theta_1-\mu_1)+\kappa_2\cos(\theta_2-\mu_2)\}\end{aligned} \tag{7}$$

  where $F_{\Theta_1}(\theta_1)$ and $F_{\Theta_2}(\theta_2)$ are the distribution functions of Θ_1 and Θ_2 respectively.

- **Example 2:** Let $f_{\Theta_1}(.)$ be the von Mises density with parameters μ_1 and κ_1 and $f_{\Theta_2}(.)$ be the von Mises density with parameters μ_2 and κ_2. Let us take the joining density $g(.)$ as the circular uniform density. Then assuming that the joint density of Θ_1 and Θ_2 admits the representation in (1), the circular-circular density with marginals $f_{\Theta_1}(.)$ and $f_{\Theta_2}(.)$ is given by

$$\begin{aligned} f_{\Theta_1,\Theta_2}(\theta_1,\theta_2) = \frac{1}{2\pi I_0(\kappa_1) I_0(\kappa_2)} &\exp\{\kappa_1 \cos(\theta_1 - \mu_1) + \\ &\kappa_2 \cos(\theta_2 - \mu_2)\} \end{aligned} \tag{8}$$

Another advantage of expressing the joint density in terms of copula function is that it facilitates the simulation of random variables from the density. Following the procedure of random variate generation from the density (2) as discussed in Jones et al. (2015), random variables are generated from the densities in (7) and (8) as mentioned below:

- **Step 1.** Θ_1 is generated from the circular uniform distribution, followed by generation of Ω from von Mises distribution with parameters μ and κ for the density in (7) or from the circular uniform distribution for the density in (8).
- **Step 2.** Set $\Theta_2 = (\Omega - \Theta_1) \,(\text{mod } 2\pi)$. Θ_1, Θ_2 thus generated are random variates from the circula density 'c'.
- **Step 3.** Given (Θ_1, Θ_2) generated in Step 2, $\Theta_1^* = F_{\Theta_1}^{-1}(\Theta_1/2\pi) \,(\text{mod } 2\pi)$, $\Theta_2^* = F_{\Theta_2}^{-1}(\Theta_1/2\pi) \,(\text{mod } 2\pi)$ are random variables from the densities in (7) and (8), where $F_{\Theta_1}^{-1}(.)$ and $F_{\Theta_2}^{-1}(.)$ are the inverse of the distribution functions $F_{\Theta_1}(\theta_1)$ and $F_{\Theta_2}(\theta_2)$ of θ_1 and θ_2 respectively.

Figure (1(a)) and (1(b)) show the contour plots of the densities in (7) and (8) respectively. The efficiency of the proposed algorithm in estimating the joint circular-circular density can be adjudged by computing the Mean Square Error (MSE) of the estimator which is given by the formula

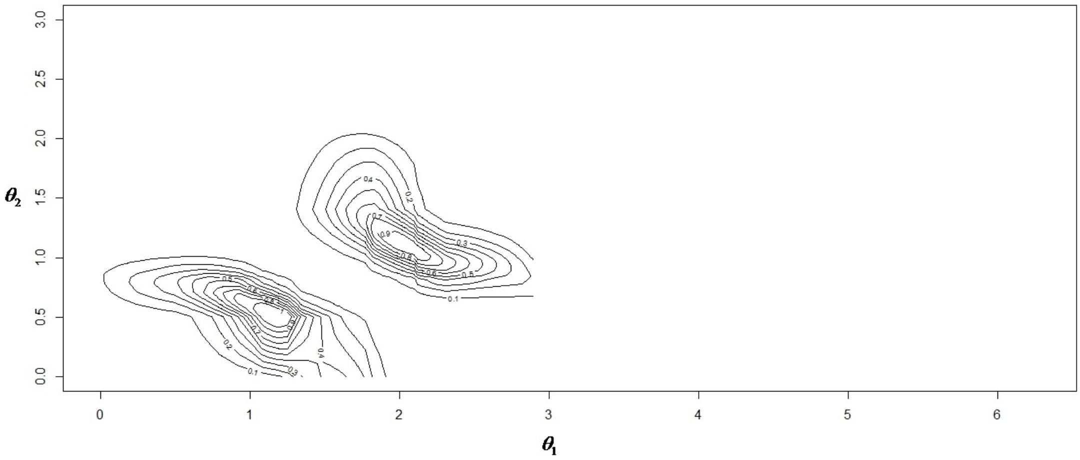
$$E\left\{\hat{f}(\theta_1,\theta_2) - f(\theta_1,\theta_2)\right\}^2$$

where $\hat{f}(\theta_1,\theta_2)$ is the estimated density and $f(\theta_1,\theta_2)$ is the true density of (θ_1,θ_2), the expectation being with respect to $f(\theta_1,\theta_2)$.

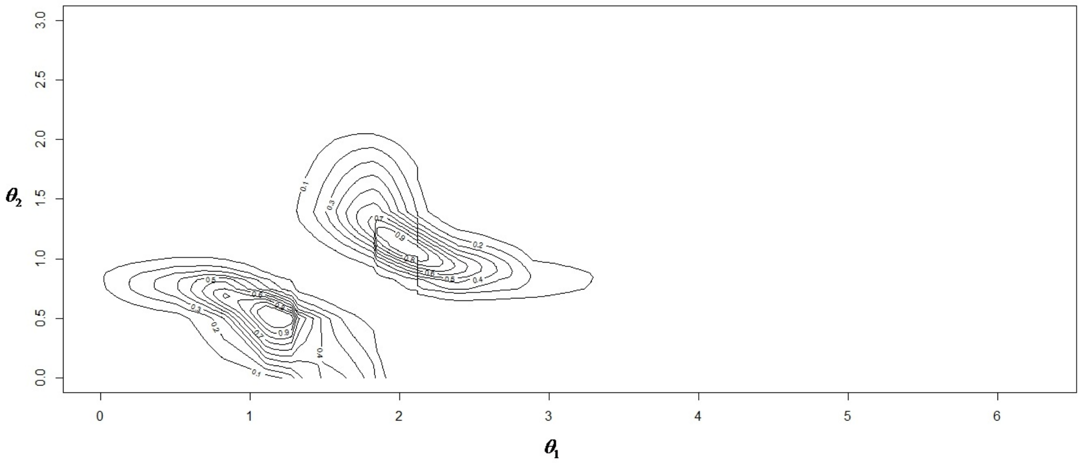
The above expectation is approximated with the help of the Monte Carlo technique, by taking 1000 replicates.

The simulation study, performance analysis of the proposed algorithm and the real-life data analysis have been carried out using the **R** software, version 3.3.0, through the user-contributed packages viz. *CircStats* (Lund and Agostinelli, 2012) and *circular* (Lund and Agostinelli, 2013) with the help of self-programmed codes.

As far as the maximum likelihood estimation of the parameters under the existing method of joint density estimation is concerned, the *Optim* function of the **R** software, version 3.3.0 has been used, together with L-BFGS-B implementation of the optimisation method.



(a) Contour plot of the density in equation (7) with parameters $\mu_1 = \pi/2$, $\kappa_1 = 3$, $\mu_2 = \pi/4$, $\kappa_2 = 5$, $\mu = \pi$ and $\kappa = 2$



(b) Contour plot of the density in equation (8) with parameters $\mu_1 = \pi/2$, $\kappa_1 = 3$, $\mu_2 = \pi/4$ and $\kappa_2 = 5$

Figure 1: Contour plots

4.1 Performance measures of the proposed algorithm for Example 1 and Example 2

The MSE of the density estimates for five sample sizes have been computed under both the examples viz. n = 10, 100, 250, 500 and 1000. The set up parameters for the first example are: $\mu_1 = \pi/2$, $\kappa_1 = 3$, $\mu_2 = \pi/4$, $\kappa_2 = 5$, $\mu = \pi$ and $\kappa = 2$. The set up parameters for the second example are $\mu_1 = \pi/2$, $\kappa_1 = 3$, $\mu_2 = \pi/4$ and $\kappa_2 = 5$. Tables (1) and (2) display the MSE values for estimating the joint circular-circular density by the proposed algorithm for example 1 and example 2 respectively: It can be seen from

Table 1: MSE values for estimating the joint circular-circular density by the proposed algorithm for example 1

n	MSE
10	5.994×10^{-2}
100	1.675×10^{-2}
250	6.784×10^{-3}
500	3.715×10^{-3}
1000	2.115×10^{-4}

Table 2: MSE values for estimating the joint circular-circular density by the proposed algorithm for example 2

n	MSE
10	9.578×10^{-2}
100	3.679×10^{-2}
250	7.136×10^{-3}
500	4.557×10^{-3}
1000	6.991×10^{-4}

tables (1) and (2) that the MSE is reduced with an increase in the sample size. Since the order of magnitude of the MSE in each case is -2 or lower, the proposed algorithm can be believed to estimate the density values with high accuracy.

4.2 Performance measures of the existing method of density estimation for Example 1 and Example 2

The MSE of the density estimates using the existing estimation method for five sample sizes have been computed under both the examples viz. n = 10, 100, 250, 500 and 1000. The set up parameters considered for the first and second example are the same as considered under section (4.1), except that the mean direction parameter is taken as 0 for both the cases. Tables (3) and (4) display the MSE values for estimating the joint circular-circular density by the existing estimation method for example 1 and example 2 respectively:

It is observed from tables (3) and (4) that the MSE values for the joint density estimates calculated using the existing algorithm are higher in comparison to those calculated

Table 3: MSE values for estimating the joint circular-circular density by the existing algorithm for example 1

n	MSE
10	7.467×10^{-1}
100	2.315×10^{-1}
250	9.146×10^{-2}
500	2.093×10^{-2}
1000	1.283×10^{-2}

Table 4: MSE values for estimating the joint circular-circular density by the existing algorithm for example 2

n	MSE
10	8.418×10^{-1}
100	3.092×10^{-1}
250	1.931×10^{-1}
500	6.785×10^{-2}
1000	2.351×10^{-2}

using the proposed algorithm. This shows that the proposed algorithm is more efficient in estimating the joint circular-circular density estimates.

5 Applications

In this section, the performance measures of the proposed algorithm for both Example 1 and Example 2 are reported. The joint density of a real-life circular-circular data set has been estimated using the proposed algorithm. The data set (Data set-I) refers to the measurements of wind directions at 6.00 am and 12.00 noon, on each of 21 consecutive days, at a weather station in Milwaukee, published in (Fisher (1993), Appendix B.21) which has been procured from Johnson and Wehrly (1977).

5.1 Joint density estimation of the data set-I

The wind directions measured at 6.00 am and 12.00 noon, on each of 21 consecutive days, at a weather station in Milwaukee, arise in terms of angles and so, are circular random variables. Consequently, the joint density of these wind directions can be estimated using the proposed algorithm. The marginal densities of the wind direction measured at

6.00 am (θ_1) and that measured at 12.00 noon (θ_2) have been ascertained with the aid of goodness-of-fit test for circular distribution (Bhattacharjee and Das, 2017).
It is found that the von Mises distribution is the best fitting distribution to both θ_1 and θ_2. The m.l.e of the parameters μ and κ of $vM(\mu, \kappa)$ are (Jammalamadaka and SenGupta, 2001):

$$\hat{\mu} = \bar{\theta}, \hat{\kappa} = A^{-1}\left[\frac{I_1(\kappa)}{I_0(\kappa)}\right] = A^{-1}\left[\frac{1}{n}\sum_{i=1}^{n}\cos(\theta_i - \mu)\right]$$

where $\bar{\theta}$ is the mean direction of the sample $\theta_1, \theta_2, \ldots, \theta_n$ and A^{-1} is the inverse function of the ratio of the first and zeroth order Bessel functions of the first kind.
Table (5) enlists the maximum likelihood estimate (m.l.e) of the parameters of the best fitting marginal densities of θ_1 and θ_2 and the p- value of the goodness-of-fit test.

Table 5: m.l.e of parameters of the best fitting marginal densities of θ_1 and θ_2 and p-value of the goodness-of-fit test

Variable	Best fitting distribution	m.l.e of parameters	p-value
θ_1	von Mises (μ_1, κ_1)	$\hat{\mu}_1 = -1.105$, $\hat{\kappa}_1 = 0.504$	> 0.05
θ_2	von Mises (μ_2, κ_2)	$\hat{\mu}_2 = 0.971$, $\hat{\kappa}_2 = 0.217$	> 0.05

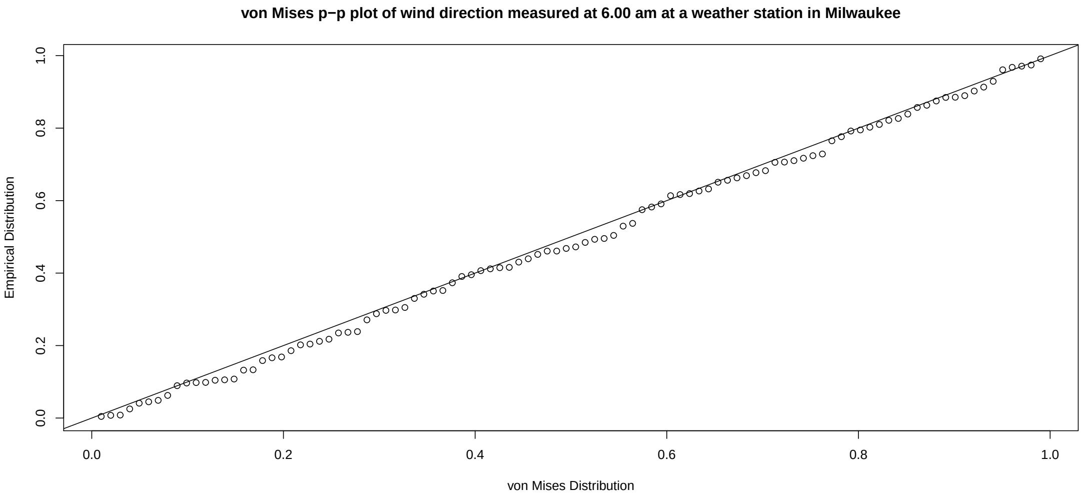
Figure (2(a)) and (2(b)) contain the p-p plot of the best fitting von Mises distribution to θ_1 and θ_2 respectively.
Further, θ_1 and θ_2 are positively associated, i.e., there exists the following relation between θ_1 and θ_2 (Fisher, 1993):

$$\theta_2 = \theta_1 + \theta_0 \,(\mod 2\pi)$$

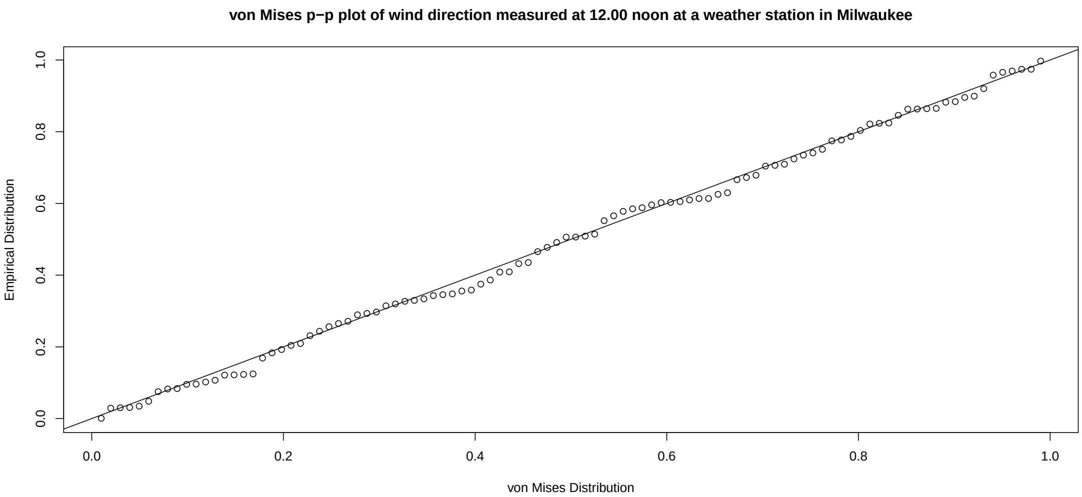
Therefore, the joining density of θ_1 and θ_2, given by $2\pi\left\{F_{\Theta_1}(\theta_1) + F_{\Theta_2}(\theta_2)\right\}$ which is not uniform, is estimated. The joining density is found to follow von Mises distribution with m.l.e of the parameters as $\hat{\mu} = 0.208$ and $\hat{\kappa} = 0.210$ (p-value of the goodness-of-fit test > 0.05). Figure (3) shows the p-p plot of the von Mises distribution fitted to the joining density of θ_1 and θ_2. Finally, the joint density of θ_1 and θ_2 is estimated with the help of the formula given in equation . The estimated joint density obtained from (7) is given by

$$\begin{aligned}\hat{f}_{\Theta_1,\Theta_2}(\theta_1, \theta_2) = & \frac{1}{2\pi I_0(0.210) I_0(0.504) I_0(0.217)} \exp\{0.210\cos[2\pi F_{\Theta_1}(\theta_1) + \\ & 2\pi F_{\Theta_2}(\theta_2) - 0.208] + 0.504\cos(\theta_1 + 1.105) + 0.217\cos(\theta_2 - \\ & 0.971)\}\end{aligned} \tag{9}$$

Figure (4) shows the contour plot of θ_1 and θ_2.



(a) p-p plot of the best fitting von Mises distribution to θ_1



(b) p-p plot of the best fitting von Mises distribution to θ_2

Figure 2: p-p plots

6 Discussion

Circular-circular distributions are studied to assess the joint effect of two circular random variables. This paper thrives to propose an algorithm for the estimation of the joint density of two circular random variables, the form of which is as given in Downs (1974). The expression of the density in terms of circulas copula, known as the circula aided in simulating from the distribution. The performance analysis of the algorithm showed the method to be quite efficient, even in case of small samples. Also, the proposed algorithm

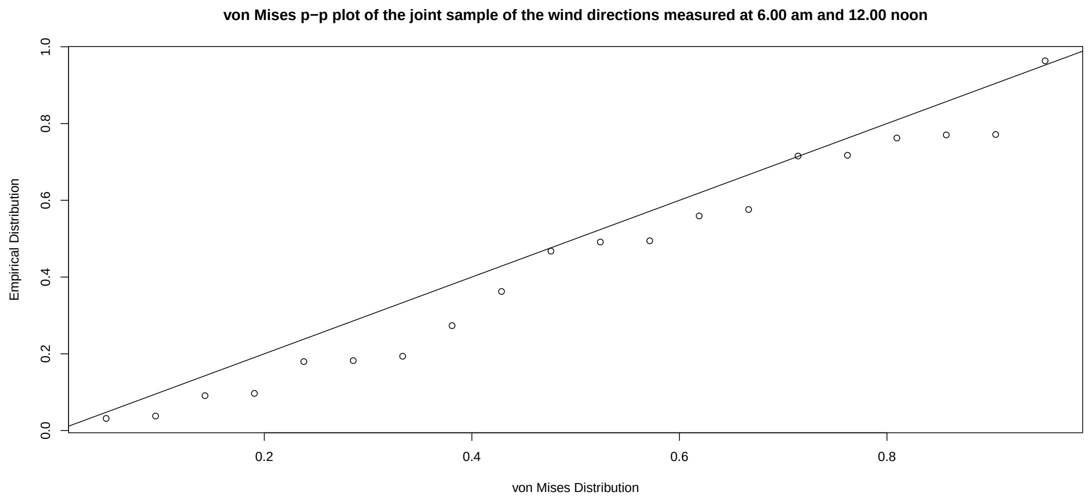


Figure 3: p-p plot of the von Mises distribution fitted to the joint sample of θ_1 and θ_2

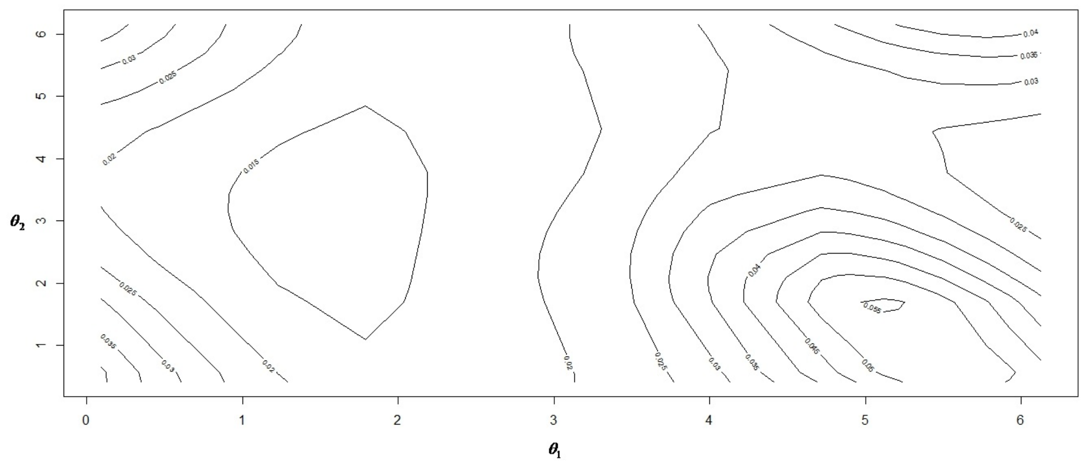


Figure 4: Joint density plot of θ_1 and θ_2

gives lower mean square error values for the joint density estimates in comparison to that provided by an existing method of density estimation, which shows its efficiency over the existing method. Further, the joining circular density estimate obtained during estimation by the proposed algorithm enables us to assess the dependence between the two circular variables, without having to calculate a coefficient of association separately. Finally, the joint density of a real-life circular-circular data set has been estimated using the algorithm. In case the form of the marginal densities are difficult to be estimated, the non-parametric kernel density estimation methods can be taken resort to.

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