Electronic Journal of Applied Statistical Analysis
EJASA, Electron. J. App. Stat. Anal.
http://siba-ese.unisalento.it/index.php/ejasa/index
e-ISSN: 2070-5948
DOI: 10.1285/i20705948v10n1p294

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Published: 26 April 2017

Electronic Journal of Applied Statistical Analysis
Vol. 10, Issue 01, April 2017, 294-306
DOI: 10.1285/i20705948v10n1p294

A Bayesian approach to estimate the reliability $P(X > Y)$ utilizing an initial guess

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Published: 26 April 2017

This paper presents a Bayesian approach to estimate the probability that one random variable exceeds another based on two independent Weibull-distributed random variables. The proposed methodology utilizes an initial guess of the reliability through an informative prior, which constitutes the cornerstone of the model. A Monte Carlo simulation study is conducted to compare the performance of the new estimators with both the Maximum Likelihood Estimation (MLE) and the Shrinkage Estimation methods. The comparison is conducted with respect to the Mean Squared Error (MSE) for different values of the scale and shape parameters of the Weibull distribution using small, moderate and large sample sizes. The proposed method outperforms the two aforementioned alternative methods. The proposed Bayesian approach is implemented using a real data regarding survival times of head and neck cancer patients for illustrative purposes

1 Introduction

Birnbaum et al. (1956) introduced the idea of estimating, $R = P(X > Y)$, the probability that one random variable exceeds another, then the idea attracted the attention of many authors in literature. R has many applications in a variety of different fields such as reliability analysis, in which R is known as the stress-strength model reliability. Furthermore, if Y models the strength of the device and X models the stress subjected on it, then the device fails any time when strength exceeded by the stress applied on it.

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ISSN: 2070-5948
http://siba-ese.unisalento.it/index.php/ejasa/index

Another interpretation of the parameter R is the effectiveness of a treatment or a drug when X and Y are the response variables for treatment and control groups respectively, see Ventura et al. (2011).

Inference on R has received a great amount of attention and it has been studied extensively in various contexts. Including; parametric and non-parametric estimation using Bayesian and frequentist methods based on different data structures. Enis and Geisser (1971), Awad et al. (1981), and Tong (1974) discussed the problem of estimating R when X and Y are either independent or bivariate exponential random variables. When X and Y are independent normal random variables, estimation of R is considered by Govindarazulu (1967); Woodward and Kelley (1977). Estimation of R is also considered by McCool (1991); Kundu and Gupta (2006); Davarzani et al. (2009) for the Weibull case. Nadarajah (2004b,a, 2005a,b) for logistic, Laplace, beta, and gamma respectively. Most recently, Samawi et al. (2016) considered the case when X and Y are dependent random variables with a bi-variate underlying distribution.

Inference on R has been investigated given different contexts of the structure of the underlying data. Muttlak et al. (2010)used ranked set sampling. Elfattah and Marwa (2008) considered the case based on censored samples. Abdel-Hady (2014) and Khamnei (2013) studied the inference on R in the presence of outliers.

In some fields, an expert often possesses some prior information of R based on either past experience or from the technical structure of the system. Given a prior estimate R_0 of R, we are looking for an estimator that incorporates this information. Thompson (1968) introduced shrinkage estimators that take advantage of the given prior guess. Baklizi and Abu Dayyeh (2003) discussed different shrinkage estimators of R when X and Y are exponential. Chaturvedi and Nandchahal (2016) studied the characteristics of the shrinkage estimators of the the Reliability of a Family of Lifetime Distributions. Haghighi and Shayib (2008) considered the shrinkage estimation of R for the Weibull type of distributions with common shape parameter. In this paper we consider the case similar to Haghighi and Shayib (2008), but from a Bayesian point of view.

Here is the organization of our paper. The elicitation of prior reflecting the previous knowledge of R together with the construction of the Bayesian model are presented in Section 2. The effectiveness of the proposed methodology is demonstrated using Monte Carlo simulation in section 3. The proposed approach is applied to a real data consisting of survival times of head and neck cancer patients in section 4.

2 Estimation Methods

A positive random variable X follows a Weibull distribution with shape parameter $\alpha > 0$ and rate parameter $\theta^{\frac{1}{\alpha}} > 0$, denoted by $W(\alpha, \theta)$, if the probability density function, $f_X(x; \alpha, \theta) = \alpha\theta x^{\alpha-1} e^{-\theta x^{\alpha}}$ for $x > 0$. The popularity of the Weibull distribution is mainly due to the versatility of its failure rate function . See Abernethy (2006) for a complete and through discussion about the advantages of Weibull distribution. In this study, let$X$ and Y have independent Weibull distributions with the same shape parameter α but with different rate parameters $\theta^{\frac{1}{\alpha}}$ and $\lambda^{\frac{1}{\alpha}}> 0$, respectively. The probability that X exceeds Y is

$$R = P\left(X > Y\right) = \int\int_{(X>Y)} f\left(x,\, y\right) dxdy = \int_0^\infty f_X\left(x\right) \left(\int_0^x f_Y\left(y\right) dy \right) dx = \frac{\lambda}{\theta + \lambda}$$

Let $\mathbf{X} = X_1,\, \ldots,\, X_n$ be a random sample from $W\left(\alpha,\, \theta\right)$ and $\mathbf{Y} = Y_1,\, \ldots,\, Y_m$ be a random sample from $W\left(\alpha,\, \lambda\right)$, then the likelihood function of the parameter vector $\left(\alpha, \theta,\, \lambda\right)$ is provided below, where $p_x = \prod_{i=1}^{n} X_i$ and $p_y = \prod_{j=1}^{m} Y_j$

$$l\left(\alpha,\theta, \lambda\, ; \mathbf{X},\mathbf{Y}\right) = \theta^n \lambda^m \alpha^{n+m} \left(p_x p_y\right)^{\alpha-1} e^{-\left(\theta \sum_{i=1}^{n} X_i^\alpha + \lambda \sum_{j=1}^{m} y_j^\alpha\right)}$$

2.1 Maximum Likelihood Estimation (MLE)

The MLE estimator of R discussed in Haghighi and Shayib (2008) is $\hat{R}_{MLE} = \frac{\hat{\lambda}}{\hat{\theta}+\hat{\lambda}}$, where $\hat{\lambda} = \frac{\sum_{j=1}^{m} y_i^\alpha}{m}$ and $\hat{\theta} = \frac{\sum_{i=1}^{n} x_i^\alpha}{n}$. For comparative purposes the shape parameter α is assumed to be known.

2.2 Shrinkage Estimation

Given a the initial guess of R to be R_0, then the Shrinkage estimator of R is $\hat{R}_{SHR} = w\hat{R}_{MLE} + \left(1 - w\right) R_0$, where $0 \leq w \leq 1$. The best method, based on Haghighi and Shayib (2008), is to choose θ such that the mean square error of $\hat{R}_{SHR}$ is minimized. The aforementioned paper provides explicit formulas to obtain $\hat{R}_{SHR}$. For comparative purposes the shape parameter α is assumed to be known.

2.3 Bayesian Estimation

This section presents the building blocks of the Bayesian model; Prior, Likelihood, and posterior. The Bayesian estimators are developed using the squared error loss function, so the posterior mean is used to estimate each parameter of interest. Non-informative priors are used, when appropriate, so that posterior densities would be dominated by the sample data, Gelman et al. (2014) .

2.3.1 Likelihood

The parameterization of the likelihood $l\left(\alpha,\theta, \lambda\, ; \mathbf{X},\mathbf{Y}\right)$ is modified to facilitate the estimation of R. Define $S = \theta + \lambda$ and $R = \frac{\lambda}{\theta+\lambda}$, then the new likelihood function of the parameter vector $\left(\alpha,\, R,\, S\right)$ is

$$l\left(\alpha, \mathrm{R}, \mathrm{S}\, ; \mathbf{X},\mathbf{Y}\right) = \alpha^{n+m} \left(p_x p_y\right)^{\alpha-1} S^{n+m+1} \left(1 - R\right)^n R^m e^{-\left((1-R)S \sum_{i=1}^{n} X_i^\alpha + RS \sum_{j=1}^{m} y_j^\alpha\right)}$$

2.3.2 Prior

The parameters $(\alpha, \mathrm{R}, \mathrm{S})$ are assumed to be priori independent, so $\pi(\alpha, \mathrm{R}, \mathrm{S}) = \pi(\alpha)\,\pi(\mathrm{R})\,\pi(\mathrm{S})$. Non-informative priors are widely used through the Jeffery's prior Jeffreys (1961). The individual priors are considered below:

- Jeffery's prior$\pi_J(S)$ is assigned to the parameter S. So,$\pi_J(S) \propto \sqrt{I(S)}$, where $I(S) = -E_X\left[\frac{\partial^2 logL}{\partial S^2}\right]$ and L is the log-likelihood function per observation given by $L = log(S) - ((1-R)\,x^\alpha + Ry^\alpha)\,S$. Differentiating L twice with respect to S, implies $\frac{\partial^2 logL}{\partial S^2} = \frac{-2}{S^2}$. Therefore $\pi_J(S) \propto \sqrt{I(S)} = \frac{1}{S}$.

- Jeffery's prior$\pi_J(\alpha)$ is assigned to the parameter α. So, $\pi_J(\alpha) \propto \sqrt{I(\alpha)}$, where $I(\alpha) = -E_X\left[\frac{\partial^2 logL}{\partial \alpha^2}\right]$, and L is the log-likelihood function per observation given by $L = 2log(\alpha) + (\alpha - 1)\,log(xy) - (1-R)\,Sx^\alpha - RSy^\alpha$. Differentiating L twice with respect to α implies $\frac{\partial^2 logL}{\partial \alpha^2} = \frac{-2}{\alpha^2} - (1-R)\,S\,log^2(x)\;x^\alpha - RS\,log^2(y)\;y^\alpha$. Notice that, $E_X\left[(1-R)\,S\,log^2(x)\;x^\alpha\right] = \left(\frac{(1-R)S}{\alpha}\right)^2 \int\limits_0^\infty log^2(u)\,ue^{-(1-R)Su}du \propto \frac{\text{Constant}}{\alpha^2}$, and $E_X\left[RS\,log^2(x)\;x^\alpha\right] = \left(\frac{RS}{\alpha}\right)^2 \int\limits_0^\infty log^2(u)\,ue^{-RSu}du \propto \frac{\text{Constant}}{\alpha^2}$. Since the both of the aforementioned integrals are convergent, then $\pi_J(\alpha) \propto \sqrt{I(\alpha)} = \frac{1}{\alpha}$.

- R is assumed to follow a proper informative prior which incorporates the initial guess for the reliability value R_0. The prior information about R is expressed in terms of the Beta distribution density $\pi(R;\, A, B) \propto (1-R)^{A-1}\,R^{B-1}$. The hyper-parameters A and B are carefully determined such that:

  1. The prior $\pi(R;\, A, B)$ should assign a considerable amount of density on the initial guess R_0. Therefore the prior mean of R is set to be R_0.

  2. The impact of the initial guess value, R_0, on the properties of $\hat{R}$ is reflected through, σ_R^2, the prior variance of R. Small values of σ_R^2 implies prominent influence of the initial guess on $\hat{R}$, which is preferred with small sample size. On the other hand, when sample size is large, then large values of σ_R^2 allow the data to dominate the value of $\hat{R}$.

Therefore, given both R_0 and σ_R^2. Set $R_0 = \frac{A}{A+B}$ and $\sigma_R^2 = \frac{AB}{(A+B)^2(A+B+1)}$, then solve for A and B, as follows:

$$A = \frac{R_0^2(1-R_0)}{\sigma_R^2} - R_0 \qquad (1)$$

$$B = \frac{R_0(1-R_0)^2}{\sigma_R^2} - (1-R_0) \qquad (2)$$

2.3.3 Full Conditional Posterior Densities:

The joint posterior distribution is

$$\pi\left(\alpha,\mathrm{R},\mathrm{S}\right|\mathbf{X},\mathbf{Y}) \propto \alpha^{n+m-1}\left(p_x p_y\right)^{\alpha-1} S^{n+m}\left(1-R\right)^{A+n} R^{B+m} e^{-\left((1-R)S\sum\limits_{i=1}^{n} X_i^{\alpha}+RS\sum\limits_{j=1}^{m} y_j^{\alpha}\right)}$$

Clearly, the joint posterior distribution of the parameters of interest is analytically intractable. So, the Gibbs sampling together with MCMC methods are employed to obtain the posterior distribution of $(\alpha,\mathrm{R},\mathrm{S})$. Straightforward algebra shows that the full posterior distributions conditional on the values of the others are shown below

1. $\pi\left(S\,|\,\mathbf{.}\right) \propto S^{n+m} e^{-\left((1-R)\sum\limits_{i=1}^{n} X_i^{\alpha}+R\sum\limits_{j=1}^{m} y_j^{\alpha}\right)S}$, which is Gamma distribution with shape $n+m+1$ and rate $(1-R)\sum\limits_{i=1}^{n} X_i^{\alpha}+R\sum\limits_{j=1}^{m} y_j^{\alpha}$.

2. $\pi\left(\alpha\,|\,\mathbf{.}\right) \propto \alpha^{n+m-1}\left(p_x p_y\right)^{\alpha-1} e^{-\left((1-R)S\sum\limits_{i=1}^{n} X_i^{\alpha}+RS\sum\limits_{j=1}^{m} y_j^{\alpha}\right)}$.

3. $\pi\left(R\,|\,\mathbf{.}\right) \propto \left(1-R\right)^{A+n} R^{B+m} e^{\left(\sum\limits_{i=1}^{n} X_i^{\alpha}-\sum\limits_{j=1}^{m} y_j^{\alpha}\right)RS}$.

Notice that the conditional posterior densities $\pi\left(\alpha\,|\,\mathbf{.}\right)$ and $\pi\left(R\,|\,\mathbf{.}\right)$ are uni-modal log-concave functions. Therefore, we can use the adaptive rejection sampling algorithm proposed by Gilks and Wild (1992) to sample directly from the their full conditional distributions.

Define the Bayesian estimator to be posterior mean of$R$ given by, $\hat{R}_{BAY} = E\left(R\,|\,\mathbf{X},\mathbf{Y}\right)$. The proposed estimator, $\hat{R}_{BAY}$, is compared with both alternative estimators, $\hat{R}_{MLE}$ and $\hat{R}_{SHR}$, through a simulation study provided in the following section.

3 Simulation Study:

3.1 Simulation Setting

A simulation study is conducted to investigate, based on empirical evidence, the performance of $\hat{R}_{BAY}$ compared with both $\hat{R}_{MLE}$ and $\hat{R}_{SHR}$. Different scenarios, see Table 1, of the parameters of the underlying Weibull distributions are considered to observe the effect on the performance. α is assigned the value 2 for the simulation. At each scenario, 1000 data sets are simulated from the true Weibull distributions. The Bayesian estimators, $\hat{R}_{BAY}$, is obtained from each simulated dataset with an MCMC of length 1000 iterations such that 600 burn-in, while the last 400 draws as the posterior sample from the target joint posterior distribution, $\pi\left(\mathrm{R},\mathrm{S}\right|\mathbf{X},\mathbf{Y})$. The sample sizes m and n are assumed to be equal and they are set to equal 5, 10, 15 for each scenario. Since the sample sizes are small, then the prior variance of R should be set to a small value to have a considerable impact of the initial guess on $\hat{R}_{BAY}$. Therefore $\sigma_R = 0.02$.

Table 1: Simulation Scenario, Parameter Values

Scenario	R	S	λ	θ	Initial Guess R_0	Prior Parameters for R (A, B)Values for Beta-Distribution
I	0.5	8	4	4	0.45	(277, 339)
II	0.2	10	2	8	0.25	(350, 116)
III	0.8	10	8	2	0.75	(116, 350)

3.2 Results and Discussion

The results of the three scenarios (I, II, and III) are presented in Tables (2, 3, and 4) and Figures (1, 2, and 3), respectively. The tables and the figures are designed to highlight two main aspects:

- Aspect One: Understand the effect of varying the initial guess on the performance of the three estimators, which is provided in Figures 1, 2, and 3.
- Aspect Two: Understand the effect of varying the sample size on the performance of each estimators, which is summarized in Tables 2, 3, and 4.

3.2.1 Aspect One:

The relative efficiency of $\hat{R}_{BAY}$ with respect to both $\hat{R}_{SHR}$ and $\hat{R}_{MLE}$ are presented, for each scenario, in Figures 1, 2, and 3, at a sample size=10. In each figure, the solid curve is $\frac{MSE(\hat{R}_{BAY})}{MSE(\hat{R}_{SHR})}$ while the dashed curve is $\frac{MSE(\hat{R}_{BAY})}{MSE(\hat{R}_{MLE})}$. The two curves are presented as functions of the initial guess R_0. As the value of the relative efficiency curve exceeds 1, it implies the out-performance of $\hat{R}_{BAY}$.

1. As long as the initial guess is within 0.1 of the true value, $MSE\left(\hat{R}_{BAY}\right) < MSE\left(\hat{R}_{MLE}\right)$ regardless of the scenario. On the other hand, as long as the initial guess is within 0.05 of the true value, $MSE\left(\hat{R}_{BAY}\right) < MSE\left(\hat{R}_{SHR}\right)$ especially if $R \geq 0.5$.

2. A cursory examination of Figures, especially Figure 2, shows the out-performance $\hat{R}_{BAY}$ over $\hat{R}_{SHR}$ in the region where the initial guess is about 0.05 above the true value i.e. when $0 < R_0 - R < 0.05$. In other words, if there is a belief that the initial guess is above R then $\hat{R}_{BAY}$ should be preferred over $\hat{R}_{SHR}$.

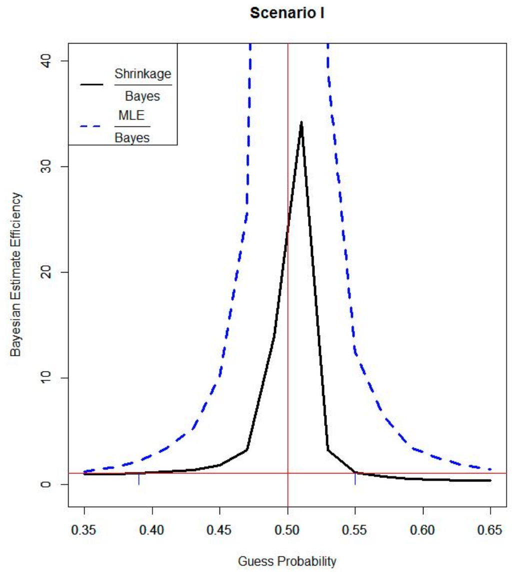

Figure 1: Bayesian Estimate Efficiency for Scenario I

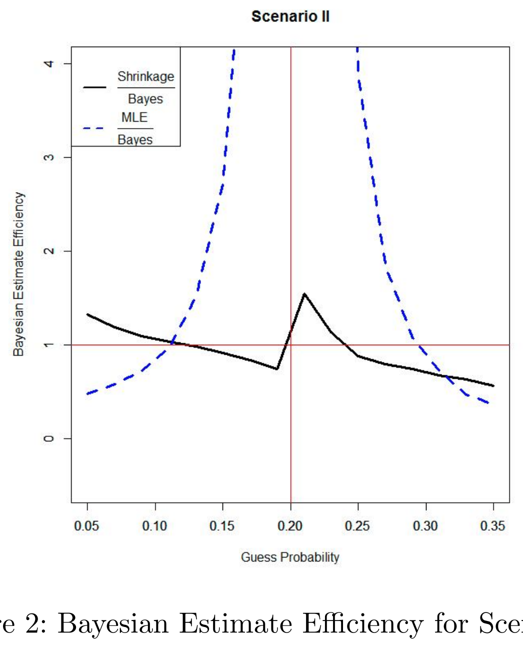

Figure 2: Bayesian Estimate Efficiency for Scenario II

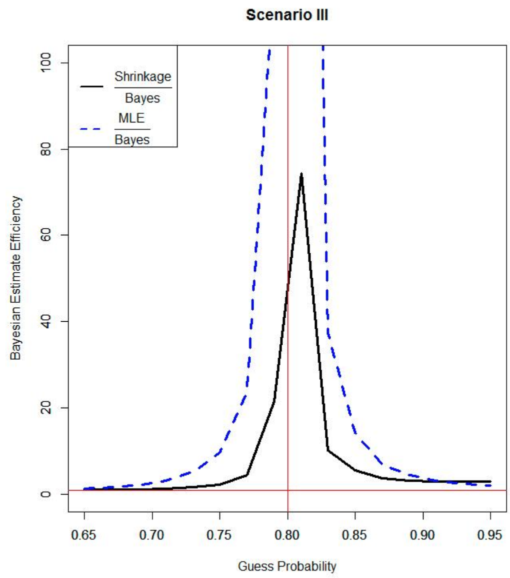

Figure 3: Bayesian Estimate Efficiency for Scenario III

3.2.2 Aspect Two:

The mean square error for any estimator $\hat{R}$ of is $MSE\left(\hat{R}\right) = E\left[\left(\hat{R} - R\right)^2\right] = \underbrace{E\left[\left(\hat{R} - E\left[\hat{R}\right]\right)^2\right]}_{Var(\hat{R})} + \underbrace{\left[E\left(\hat{R}\right) - R\right]^2}_{Bias(\hat{R})}$. Table 2 presents the $E\left[\hat{R}\right]$, standard deviation $\sqrt{Var\left(\hat{R}\right)}$, and the mean square error $MSE\left(\hat{R}\right)$, respectively, at each sample size for all the considered scenarios.

1. As generally expected, the mean square error for each estimator decrease as sample size increases, especially $\hat{R}_{MLE}$.

2. As ample size decreases, the impact of the initial guess, on reducing the MSE of both $\hat{R}_{BAY}$ and $\hat{R}_{SHR}$, is more obvious. This demonstrates the intuition behind preferring such estimators over MLE especially when sample size is small.

3. Notice that for all sample sizes, within each scenario, $MSE(\hat{R}_{BAY}) \leq MSE(\hat{R}_{SHR}) < MSE(\hat{R}_{MLE})$. Furthermore, $\sqrt{Var\left(\hat{R}_{BAY}\right)}$ is the minimum in all cases, which implies that $\hat{R}_{BAY}$ is more efficient estimator of R.

Table 2: Results for Scenario I

Sample size		$\hat{R}_{BAY}$	$\hat{R}_{SHR}$	$\hat{R}_{MLE}$
5	$E\left[\hat{R}\right]$	0.451	0.449	0.516
	$\sqrt{Var\left(\hat{R}\right)},$	0.003	0.012	0.167
	$MSE\left(\hat{R}\right)$	0.002	0.003	0.028
10	$E\left[\hat{R}\right]$	0.451	0.454	0.502
	$\sqrt{Var\left(\hat{R}\right)}$	0.003	0.015	0.115
	$MSE\left(\hat{R}\right)$	0.002	0.002	0.013
15	$E\left[\hat{R}\right]$	0.452	0.460	0.504
	$\sqrt{Var\left(\hat{R}\right)}$	0.004	0.196	0.097
	$MSE\left(\hat{R}\right)$	0.002	0.002	0.009

Table 3: Results for Scenario II

Sample size	$\hat{R}_{BAY}$	$\hat{R}_{SHR}$	$\hat{R}_{MLE}$
5	0.248	0.247	0.182
	0.005	0.009	0.11
	0.002	0.002	0.011
10	0.247	0.25	0.223
	0.005	0.016	0.085
	0.002	0.003	0.008
15	0.247	0.246	0.203
	0.006	0.010	0.065
	0.002	0.002	0.004

Table 4: Results for Scenario III

Sample size	$\hat{R}_{BAY}$	$\hat{R}_{SHR}$	$\hat{R}_{MLE}$
5	0.749	0.736	0.662
	0.004	0.044	0.143
	0.003	0.006	0.039
10	0.749	0.733	0.654
	0.005	0.058	0.110
	0.003	0.008	0.033
15	0.751	0.741	0.678
	0.007	0.052	0.090
	0.002	0.006	0.023

Table 5: Survival Times of Two Treatment Groups of Head and Neck Cancer Patients

Radio (X)	6.53,7,10.42,14.48,16.1,22.7,34,41.55,42,45.28,49.4,53.62,63 64,83,84,91,108,112,129,133,133,139,140,140,146,149,154 157,160,160,165,146,149,154,157,160,160,165,173, 1146,1417 176,218,225,241,248,273,277,297,405,417,420,440,523,583,594,1101,
Radio & Chemo (Y)	12.2,23.56,23.74,25.87,31.98,37,41.35,47.38,55.46,58.36,63.47,68.46,78.26, 74.47,81,43,84,92,94,110,112,119,127,130,133,140,146,155,159,173, 179,194,195,209,249,281,319,339,432,469,519,633,725,817,1776

4 Real Data Analysis

4.1 Data Description

To illustrate our method on real data, we consider a data presented in Singh et al. (2015), which consists of the survival times of two treatment groups of head and neck cancer patients from . The first group represents the survival times of 58 head and neck cancer patients treated with radiotherapy, while the other group represents the survival times of 45 head and neck cancer patients treated with combined radiotherapy and chemotherapy. The survival times are provided in Table 5.

4.2 Data Analysis Results

In this section we analyzed 1000 bootstrap samples that were randomly selected with replacement from the above two groups such that $n = 29$ and $m = 23$. The true value of R is set to be 0.5, using the findings in Singh et al. (2015), based on analyzing the complete data. The common shape parameter of the two underlying distributions is $\alpha = 0.97$. Table 6 presents the mean square error of the three estimators using four different initial guess values. Clearly, $\hat{R}_{BAY}$ outperforms both $\hat{R}_{BAY}$ and $\hat{R}_{SHR}$ especially when

Table 6: MSE of R Using the Three Methods

Initial Guess R_0	MSE$\left(\hat{R}_{BAY}\right)$	MSE $\left(\hat{R}_{SHR}\right)$	MSE$\left(\hat{R}_{MLE}\right)$
0.45	0.0021	0.0014	0.0059
0.48	0.0003	0.0004	0.0047
0.53	0.0008	0.0010	0.0055
0.55	0.0022	0.0019	0.0062

the initial guess is about 0.05 within the true value i.e. when $|R_0 - R| < 0.05$. This result agrees with findings of the simulation results in section 3.2.

5 Conclusion

The present study proposes a new approach that utilizes an initial guess of the reliability through an informative prior shrinkage estimation of R based Weibull distributions with common shape parameter. The results of the simulation study demonstrate that the Bayesian estimator outperforms the existing shrinkage estimators as long as the initial guess is about 0.05 above the true value, regardless of the underlying distribution parameters. The use of the Bayesian estimator is worth considering especially if available sample size is small. As generally expected, as sample size increases, the precision of MLE estimator increases, while both Bayesian and shrinkage estimators are still affected by the prior guess.

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