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By Mamun et al.

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The asymptotic confidence interval of the slope in linear structural relationship model is usually used to draw the inference about parameter. It is now evident that, asymptotic inference is often unreliable for small-sample. In small samples, asymptotic inference may be unreliable as standard errors may be imprecise, leading to incorrect confidence intervals and statistical test size. In these issues, bootstrap can be used instead of asymptotic inference to deal with these challenging problems. We consider both the parametric and the jackknife-after-bootstrap methods for this particular study. The performances of both confidence intervals are studied by real world data and Monte Carlo simulations. Our findings show that overall the bootstrap confidence intervals perform better than the asymptotic confidence interval for small samples in terms of coverage probability with reasonable expected length.

keywords: linear structural relationship model, asymptotic confidence interval, bootstrap confidence interval, coverage probability, expected length.

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1 Introduction

In linear regression analysis, the explanatory variables are assumed to be fixed and measured without error. But in reality, due to many practical reasons this assumption often does not exist and inherent measurement errors arise into the observations. Ignorance of measurement errors directly affects the desirable criteria of point estimators or interval estimators. A large number of literature on the errors-in-variables model (EIVM) have been developed over the years (Madansky, 1959; Moran, 1971; Kendall and Stuart, 1973; Fuller, 1987; Cheng and Van Ness, 1944).The linear structural relationship model (LSRM) is one of the families in the EIVM which also includes functional, ultrafunctional and ultrastructural relationship models. Over the past thirty years, a large number of works have been done in LSRM (Birch, 1964; Barnett, 1967; Chan and Mak, 1979; Lakshminarayanan and Gust, 1984; Reilman et al., 1985; Bolfarine and Cordani, 1993; Gillard, J.W.). However, the point estimation of LSRM parameters was the main focus in the most of the papers. A number of works has been done (Gleser and Hwang, 1987; Li, 1989; Weerahandi, 1993; Liau and Shalabh, 2009) on interval estimation of EIVM. Huwang (1996) and Tsai (2010) have addressed the interval estimation of LSRM based on the asymptotic results assuming the measurement error variance σ_δ^2 or σ_ϵ^2 is known.

Most of the works on interval estimation of LSRM are based on asymptotic results which may not hold when the sample size is small to moderate and this kind of situation is more prevalent in nature. In this situation it is difficult to find the proper distribution of the estimators and the confidence intervals are not reliable in terms of coverage probabilities and expected lengths. Efron (1979)'s bootstrap is a very useful alternative to estimate the parameters and to construct their confidence intervals when the sample size is small. Jackknife-after-bootstrap was proposed by Efron (1979) to reduce the standard error of estimators. Since each and every observation is deleted in turns, it is expected that this method is more resistant to outliers. Efron (1979) pointed out that for the jackknife-after-bootstrap, the standard error of estimators may get reduced by 80%, so this technique should help in the construction of confidence intervals.

In this article, we investigate two computer intensive confidence interval namely, parametric bootstrap confidence interval (PBCI) and jackknife-after-bootstrap confidence interval (JABCI) for the slope parameter of LSRM assuming the ratio of error variance is known and then compare these two computer intensive confidence interval with the existing traditional asymptotic confidence interval (ACI) on the basis of two criteria, viz., coverage probability and expected length. We organize this article as follows: In Section 2, we briefly discuss the parameter estimation of LSRM. The existing ACI for the slope of LSRM with known ratio of error variances is reviewed in Section 3. In Section 4, we introduce PBCI and JABCI for the slope of LSRM with known ratio of error variances. In Section 5, PBCI and JABCI are compared with ACI based on the criteria of coverage probability and expected length through the simulation study. Result of CI and length of CI for slope using two real world data set are presented in Section 6. In Section 7, we present the concluding remarks.

2 Estimation of parameters in linear structural relationship model

Consider the following circumstances

$$Y = \alpha + \beta X \tag{1}$$

where, there exists a linear relationship between the random variables X (heights) and Y (weights) and suppose that they are measured without error.

However, in reality, these two variables X and Y are not observed directly, i.e., they are measured subject to error. Assume that for each i, x_i and y_i are taken instead of X_i and Y_i respectively $(i = 1, 2, .., n)$. If δ_i and ϵ_i are the two respective errors in measuring X_i and Y_i, then we can write $x_i = X_i + \delta_i$ and $y_i = Y_i + \epsilon_i$, where the error terms δ_i and ϵ_i are normally distributed having zero mean and variance σ_δ^2 and σ_ϵ^2, respectively. This reveals that the variances of error are not dependent on i and so independent of the level of X and Y, which assumed homoscedasticity. There are some assumptions that have been described in the literature for obtaining the X values. For example, Kendall and Stuart (1973) described the structural model considering X_i as normal distribution with mean μ and variance σ_X^2. In LSRM, the errors are assumed to be normal, the bivariate normal distribution of x_i and y_i, is then

$$\begin{pmatrix} x_i \\ y_i \end{pmatrix} \sim N \left(\begin{bmatrix} \mu \\ \alpha + \beta\mu \end{bmatrix}, \begin{bmatrix} \sigma_X^2 + \sigma_\delta^2 & \beta\sigma_X^2 \\ \beta\sigma_X^2 & \beta^2\sigma_X^2 + \sigma_\epsilon^2 \end{bmatrix} \right) \tag{2}$$

Kendall and Stuart (1973) have shown that there are five equations with six unknown $(\mu, \alpha, \beta, \sigma_X^2, \sigma_\delta^2, \sigma_\epsilon^2)$, hence an additional assumption is required for the unique and consistent solutions of the parameters of the model (1). In particular, Hood et al. (1999) discuss in detail estimation procedure to estimate the model (1) under various assumptions. However, for the case when the ratio of error variance $\lambda = \frac{\sigma_\epsilon^2}{\sigma_\delta^2}$ is assumed to be known, the Maximum Likelihood Estimate (MLE) for the slope parameter β is given by

$$\hat{\beta} = \frac{(S_y^2 - \lambda S_x^2) + \sqrt{(S_y^2 - \lambda S_x^2)^2 + 4\lambda S_{xy}^2}}{2S_{xy}} \tag{3}$$

where, S_x^2, S_y^2 and S_{xy} are defined as $S_x^2 = \frac{1}{n}\sum(x_i - \bar{x})^2$, $S_y^2 = \frac{1}{n}\sum(y_i - \bar{y})^2$ and $S_{xy} = \frac{1}{n}\sum(x_i - \bar{x})(y_i - \bar{y})$, respectively.

and the variance for the slope parameter is

$$v\hat{a}r(\hat{\beta}) = \frac{\beta^2\sigma_X^2\sigma_\delta^2 + \lambda\sigma_X^2\sigma_\delta^2 + \lambda\sigma_\delta^4}{n\sigma_X^4} \tag{4}$$

3 Asymptotic Confidence Interval for the Parameter β of LSRM

In this section, the asymptotic confidence interval for LSRM with known ratio of error variances is discussed briefly. In case of identifiable model, it is convenient to use asymptotic confidence set for the interval estimation (Fuller, 1987; Cheng and Van Ness, 1944; Birch, 1964) for more details. However, only a few number of studies have been carried out on the interval estimation of LSRM on the basis of identifiability assumption. Tsai (2010) discussed the asymptotic confidence interval for the slope parameter of LSRM under the assumption of σ_δ^2 known case. The asymptotic confidence interval for the slope β of LSRM under the identifiability assumption of the ratio of two error variances known is given by Tsai (2010) as

$$\hat{\beta} \pm Z_{1-\frac{\alpha}{2}} \sqrt{v\hat{a}r(\hat{\beta})} \tag{5}$$

where $Z_{1-\frac{\alpha}{2}}$ is the $100(1-\frac{\alpha}{2})$ percentile of standard normal distribution. The estimated value of the slope is taken from (3) and $v\hat{a}r(\hat{\beta})$ is taken from (4) respectively.

4 Computer Intensive Confidence Intervals for the Slope of LSRM

The bootstrap is a computer intensive technique for measuring standard errors, biases, confidence intervals and other measures of statistical accuracy. It has become a standard statistical tool and produce accurate estimates automatically in almost any situation, including very complicated ones, without requiring much thought from the statistician. Moreover, the bootstrap method needs less assumption and easily comparable to the conventional methods. There has been a lot of theoretical and empirical research (Efron, 1979; Shao and Tu, 1996; Davison and Hinkley, 1997) examining the properties of the bootstrap estimation.

4.1 Parametric Bootstrap Confidence Interval

For parametric bootstrap (PB), first we need to select the distribution type that the data come from and find the MLE of parameters for that distribution. Then, we randomly resample the data with replacement from the fitted model. The parametric bootstrap consists of the following steps:

1. A random sample $(x_1, y_1), (x_2, y_2), ..., (x_n, y_n)$ is collected from a study of our interest.
2. The parameters $(\mu, \alpha, \beta, \sigma_X^2, \sigma_\delta^2, \sigma_\epsilon^2)$ of the distribution are determined that best fits the data from the known distribution family using maximum likelihood estimators (MLE).

Finally, we calculate the maximum likelihood estimate of the slope parameter say

$$\hat{\beta}_{PB} = \frac{1}{B} \sum_{i=1}^{B} \hat{\beta}_i$$

and standard error (se) of the slope say

$$se(\hat{\beta}_{PB}) = \sqrt{\frac{1}{B-1} \sum_{i=1}^{B} (\hat{\beta}_i - \hat{\beta}_{PB})}$$

for each bootstrap replication.

Now, the parametric bootstrap confidence interval for the slope β is given by

$$\hat{\beta}_{PB} \pm Z_{(1-\frac{\alpha}{2})} se(\hat{\beta}_{PB}) \tag{6}$$

4.2 Jackknife-After-Bootstrap Confidence Interval

The jackknife-after-bootstrap (JAB) method provides a way of estimating standard error for bootstrap quantity using only information from original bootstrap samples, i.e. further resampling is not required here. The complete procedure for JAB method consists of the following steps:

1. Leave out data point i , for $i = 1, 2, ..., n$ from the original bootstrap sample and then compute the maximum likelihood estimate of slope parameter say $\hat{\beta}_{(i)}$.

2. Compute the JAB estimate (Fox, 1997) of the slope, say $\hat{\beta}_{JAB}$ as

$$\hat{\beta}_{JAB} = \frac{1}{n} \sum_{i=1}^{n} \hat{\beta}_{(i)}$$

3. Define the standard error of JAB estimate (Wang, 1998), say as $se(\hat{\beta}_{JAB})$

$$se(\hat{\beta}_{JAB}) = \sqrt{\frac{n-1}{n} \sum_{i=1}^{n} \{se(\hat{\beta}_{(i)}) - se(\hat{\beta}_{(.)})\}^2}$$

where,

$$se(\hat{\beta}_{(i)}) = \sqrt{\frac{1}{B-1} \sum_{i=1}^{B} (\hat{\beta}_{(i)} - \hat{\beta}_{JAB})^2}$$

and

$$se(\hat{\beta}_{(.)}) = \frac{1}{B} \sum_{i=1}^{B} se(\hat{\beta}_{(i)})$$

Finally, the JAB confidence interval for β is given by

$$\hat{\beta}_{JAB} \pm Z_{(1-\frac{\alpha}{2})} se(\hat{\beta}_{JAB}) \tag{7}$$

5 Results of Simulation Study

We carried out a simulation study to investigate the performance of the ACI, PBCI and JABCI under a variety of combinations of parameters. The performance is measured under the criteria of length of CI as well as coverage probabilities based on 10,000 simulated data each replicated 200 times. The length of confidence interval for ACI, PBCI and JABCI are computed as the median of all 10,000 replications.

The simulated data set is generated by using equation (2). We choose $\mu = 10, \alpha = 0$ and $\beta = 1$ with different values of $\sigma_X^2 = 2, 5, 10$ and $\sigma_\delta^2 = 0.50, 0.75, 1.00$. For each specified set of parameters, different values of simulated data sets (x_i, y_i) are obtained. These simulated data sets are then used to estimate the slope and confidence interval of the slope of linear structural relationship model. Moreover, different sample sizes $n = 10, 30, 50$ is considered in order to investigate the performance of ACI, PBCI and JABCI.

Table 1: Results for the expected length and coverage probability of slope when $\mu = 10, \alpha = 0$, and $\beta = 1$, and significance level $\alpha = 0.05$

n	σ_X^2	σ_δ^2	Expected length			Coverage probability		
			ACI	PBCI	JABCI	ACI	PBCI	JABCI
10	2	0.50	0.8679	1.5081	1.2828	0.8805	0.9510	0.9500
		0.75	1.0783	2.4811	1.7045	0.8850	0.9520	0.9490
		1.0	1.2666	3.6733	2.1031	0.8830	0.9560	0.9450
	5	0.50	0.5323	0.7301	0.7380	0.8860	0.9430	0.9550
		0.75	0.6615	0.9584	0.9521	0.8860	0.9470	0.9520
		1.0	0.7716	1.2135	1.1358	0.8830	0.9520	0.9500
	10	0.50	0.3752	0.4755	0.5019	0.8940	0.9400	0.9510
		0.75	0.4599	0.6053	0.6282	0.8910	0.9430	0.9550
		1.0	0.5323	0.7301	0.73806	0.8860	0.9430	0.9550
30	2	0.50	0.5226	0.5392	0.6765	0.9310	0.9440	0.9510
		0.75	0.6549	0.6944	0.8747	0.9320	0.9370	0.9610
		1.0	0.7755	0.8559	1.0461	0.9300	0.9320	0.9620
	5	0.50	0.3198	0.3225	0.4093	0.9280	0.9380	0.9650
		0.75	0.3968	0.4029	0.5136	0.9290	0.9400	0.9530
		1.0	0.4611	0.4742	0.5972	0.9310	0.9430	0.9540
	10	0.50	0.2240	0.2242	0.2823	0.9300	0.9360	0.9530
		0.75	0.2760	0.2760	0.3499	0.9300	0.9370	0.9540
		1.0	0.3198	0.3225	0.4093	0.9280	0.9380	0.9540
50	2	0.50	0.4159	0.4202	0.5088	0.9420	0.9430	0.9640
		0.75	0.5226	0.5347	0.6403	0.9440	0.9460	0.9660
		1.0	0.6196	0.6427	0.7761	0.9450	0.9460	0.9650
	5	0.50	0.2532	0.2523	0.3043	0.9430	0.9430	0.9600
		0.75	0.3137	0.3148	0.3808	0.9430	0.9420	0.9600
		1.0	0.3669	0.3702	0.4495	0.9420	0.9430	0.9640
	10	0.50	0.1767	0.1752	0.2118	0.9420	0.9430	0.9620
		0.75	0.2177	0.2173	0.2605	0.9440	0.9450	0.9610
		1.0	0.2532	0.2523	0.3043	0.9420	0.9430	0.9620

From Table 1 it is shown that when the sample size is $n = 10$ and 30 the coverage probability of the jackknife-after-bootstrap is very close to our target value 0.95. Although the coverage probabilities of the parametric bootstrap are very close to those of the JABCI, the latter ones perform better on the basis of having shorter confidence intervals. The performance of the classical ACI is poor in this situation. Its coverage is roughly about 7% less than the target value. When the sample size increases to 30, the performance of the ACI improves, but still it performs less than the JABCI and the PBCI. In case of $n = 50$, the performance of the ACI improves as it possesses coverage probability close to the target value, but the PBCI produces coverage probability marginally better than the ACI which are even closer to the target value without jeopardizing the expected length. The JABCI produces coverage probabilities slightly higher than the target value. Similarly for the sample size $n = 10$ and $n = 30$, when the values of σ_X^2 and σ_δ^2 increases, the JABCI performs better than the other two methods in terms of coverage probability. But when the sample size increases to $n = 50$, the PBCI performs better than the other two methods though the coverage probability of ACI and JABCI are very close to the target value. Again when the value of σ_X^2 increase, the expected length of each method decreases and this is true for each sample size. The reverse conclusion can be drawn for σ_δ^2.

6 Numerical Examples

In this section, we consider two real world data sets to investigate the performance of three confidence intervals. In order to make the relationship as model (1), we assume that measurement error can occur in both the variables of these two examples. Firstly, the serum kanamycin data is considered which is taken form Kelly (1984) and secondly, we consider the iron in slag data taken from Hand et al. (1994). Table 2 and Table 3 demonstrate the confidence interval and the length of confidence interval for the slope parameter of ACI, PBCI and JABCI method from the serum kanamycin and iron in slag data respectively.

6.1 Serum Kanamycin Data

These data were taken from Kelly (1984) and consist of simultaneous pairs of measurements of serum kanamycin levels in blood samples drawn from twenty premature babies. One of the measurements was obtained by a heelstick method (x) and the other by using an umbilical catheter (y). Since there is a measurement error in both methods, the measurement error model seems to be appropriated for describing these data.

6.2 Iron in Slag Data

The data for the 50 results of Iron content of crushed blast-furnce slag measured by two different techniques, which are chemical test and magnetic test (Hand et al., 1994). From Table 2 and Table 3, we observe that JABCI produces the shortest confidence interval for slope followed by PBCI and ACI method.

Table 2: Results of CI and the length of CI for slope from serum kanamycin data

Method	CI	Length of CI
ACI	(0.7577, 1.3818)	0.6241
PBCI	(0.7519, 1.3415)	0.5895
JABCI	(1.0367, 1.5236)	0.4868

Table 3: Results of CI and the length of CI for slope from iron in slag data

Method	CI	Length of CI
ACI	(0.6519, 1.1920)	0.5401
PBCI	(0.6640, 1.0538)	0.3898
JABCI	(0.9152, 1.1811)	0.2659

7 Conclusions

In this article we investigate the parametric bootstrap confidence interval and jackknife-after-bootstrap confidence interval for the slope of LSRM when the ratio of the two error variances is known. Monte Carlo simulation reveals that for $n = 10$ and $n = 30$, the JABCI produces the most satisfactory results followed by the PBCI. The performance of the ACI is not satisfactory in this occasion. However, for $n = 50$, the JABCI produces coverage probabilities slightly higher than the target value. Thus it is clear that both bootstrap confidence intervals, the PBCI and the JABCI, performs well when sample size is small for the slope of linear structural relationship model.

Acknowledgement

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