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The Finite Iterative Method for calculating the correlation matrix implied by a recursive path model

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Given q+p variables (q endogenous variables and p exogenous variables) and the correlation matrix among exogenous variables, how to compute the correlation matrix implied by a given recursive path model connecting these q+p variables ? A new method called *Finite iterative method* is introduced to perform this task. The computational efficiency of this method and the well-known Jöreskog's method is discussed and illustrated.

keywords: Path Analysis, correlation matrix.

Introduction

Path Analysis (Boudon, 1965; Duncan, 1966; Heise, 1969; Hauser et al, 1975) is a set of statistical techniques to examine cause and effect between observed (measured) variables on the same set of individuals. Path Analysis is an extension of multiple regression models in the sense that variables can be both dependent and independent. This is not the case for multiple regression where only one single dependent variable is explained by several independent variables.

The origin of path analysis dates back to the beginning of the last century in the work of Sewall Wright (1921), a British geneticist who was the first to develop this multivariate statistical technique. Today, path analysis is the simplest form of structural equation models with latent variables. Path analysis is applied in many disciplines as evidenced

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by the abundant literature on this subject (Boudon, 1965; Duncan, 1966). Numerous articles deal with the use of path analysis in ecological studies (Sanchez-Pinero, F. and G.A. Polis. 2000; Shine, R. 1996; Shipley, B. 1999) and in social sciences (Blalock, 1971; Wolfle LM, 2003) .

In the vocabulary of path analysis there are two types of variables : (i) exogenous variables and (ii) endogenous variables. An exogenous variable is always an independent variable; its role is to explain other variables, which implies that it can never be an effect in the considered model, whereas an endogenous variable is a one that can be the cause (predictor) for one or more other variables and at the same time the effect of one or more other variables.

The starting point of path analysis is a conceptual diagram, considered as a schematic representation of the model; this diagram should be specified by the modeler. In addition to the causal relationships between variables, the disturbances associated to endogenous variables as well as direct effects (parameters) between variables must also be specified. When two variables are correlated and there is no causal relationship between them, this is represented by a curved arc. This diagram is represented algebraically by a set of regression equations in which at least one variable is both explanatory and explained. Figure 1 shows a path analysis model with four variables ξ_1, η_1, η_2 and η_3, together with its system of structural equations.

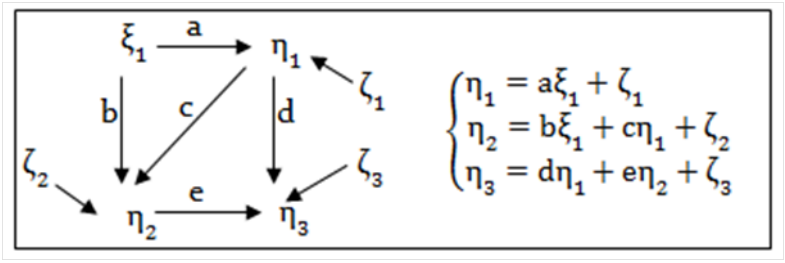

Figure 1: Representation of a path analysis model with four variables and its system of structural equations. ξ_1 is an exogenous variable (predictor), η_1, η_2 and η_3 are endogenous variables (explained), η_1 is explained by ξ_1 with direct effect noted by a, η_2 is explained by ξ_1 and η_1 with direct effects noted respectively by b and c, η_3 is explained by η_1 and η_2 with direct effects noted respectively by d and e. Disturbances are associated to endogenous variables η_1, η_2 and η_3. These disturbances are noted respectively by ζ_1, ζ_2 and ζ_3.

When a variable ξ is the cause of the variable η, much of the variance of η could be explained by ξ. The disturbance ζ associated to η represents the source of variability of η which is not explained by ξ. The disturbance is then an exogenous variable not directly measured. The magnitude of an effect is measured by a numerical quantity

called the path coefficient or parameter. It is a statistical estimate of the direct effect of an independent variable on the dependent variable taking into account other variables.

In path analysis, there are two basic kinds of models, the recursive model and the nonrecursive model. In a recursive model, the causal effects are unidirectional. In other words, no variable is both a cause and an effect of another variable, directly or indirectly. In contrast, in a nonrecursive model there is a mutual causal influence among variables.

The present paper focuses on the computation of the correlation matrix implied by a recursive path model noted $\widehat{\mathrm{R}}$. The computation of this matrix is needed at least at three different levels of the process of the analysis: (i) to test the validity of the model, (ii) to evaluate the total effect of exogenous variables, (iii) to estimate the parameters where several criteria are minimized, like $\mathrm{F} = \frac{1}{2}\mathrm{tr}\left[\left(\mathrm{R}-\widehat{\mathrm{R}}\right)\right]^2$ with R, the emprirical correlation.

The topic of the present paper can be summarized as follows. Given q+p variables (q endogenous variables and p exogenous variables) and the correlation matrix among exogenous variables, how to compute the correlation matrix implied by a given recursive path model connecting these q+p variables?

The phrase "given path model" means that the parameters of the model are known. The task for the model given in figure 1 is to provide the following correlation matrix.

$$\widehat{\mathrm{R}} = \begin{bmatrix} 1 & \mathrm{a} & \mathrm{b+ac} & \mathrm{ad+be+ace} \\ \mathrm{a} & 1 & \mathrm{ab+c} & \mathrm{d+abe+ce} \\ \mathrm{b+ac} & \mathrm{ab+c} & 1 & \mathrm{abd+cd+e} \\ \mathrm{ad+be+ace} & \mathrm{d+abe+ce} & \mathrm{abd+cd+e} & 1 \end{bmatrix}$$

The well-known method proposed by Jöreskog (1977)(see section 2 in the present paper) can be considered as a solution for the computation of the correlation matrix implied by a given recursive model. The drawback of this method is twofold : it requires matrix inversion. On the other hand, and mainly, the variances of disturbances should be computed based on model parameters. It is important to note that the expression of these variances in terms of parameters is known only for simple models. This fact constitutes the main limitation of Jöreskog's method.

In order to tackle this limitation, the present paper proposes a new method called the *Finite Iterative Method* (FIM) to compute the correlation matrix implied by a recursive path model. Unlike Jöreskog's method, the computation of the variances of disturbances using FIM is not necessary. In addition, FIM is more straightforward and can be easily implemented since matrix inversion is not needed anymore.

The paper is organized as follows. The first part presents the notations used in the rest of the paper. The second section describes Jöreskog's method as well as FIM. The third

section illustrates in examples the efficiency of FIM compared to Jöreskog's method. Finally, some conclusions will be drawn in the last section.

1 Notations

This section introduces the basic notations following Jöreskog (1977), Jöreskog and Wold (1982) and Hoyle (1995). The translation of the diagram of a recursive path model to equations is given by the following generic form :

$$\begin{bmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_q \end{bmatrix} = \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ \beta_{21} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \beta_{q1} & \cdots & \beta_{q,(q-1)} & 0 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_q \end{bmatrix} + \begin{bmatrix} \gamma_{11} & \gamma_{12} & \cdots & \gamma_{1p} \\ \gamma_{21} & \gamma_{22} & \ddots & \gamma_{2p} \\ \vdots & \ddots & \ddots & \vdots \\ \gamma_{q1} & \gamma_{q2} & \cdots & \gamma_{qp} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_p \end{bmatrix} + \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \vdots \\ \zeta_q \end{bmatrix} \quad (1)$$

System (1) can be also written in compact form as :

$$\eta = \mathrm{B}\eta + \Gamma\xi + \zeta \qquad (2)$$

where in (2) :

(i) $\eta = (\eta_1, \eta_2, \ldots, \eta_q)$ is the (q×1) vector of all endogenous variables,

(ii) $\xi = (\xi_1, \xi_2, \ldots, \xi_p)$ is the (p×1) vector of all exogenous variables,

(iii) B is the (q×q) lower triangular matrix of structural coefficients relating endogenous variables. B is always lower triangular for a recursive model.

(iv) Γ is the (q×p) matrix of structural coefficients relating endogenous variables to exogenous variables,

(v) $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_q)$ is the (q×1) vector of disturbances.

In addition, the following hypothesises are considered :

(i) the exogenous variables are standardised,

(ii) the vector of disturbances ζ is not correlated to the vector of exogenous variables ξ :

$$\begin{cases} \mathrm{E}\left(\xi\zeta^{\mathrm{t}}\right) = 0 \\ \mathrm{E}\left(\zeta\xi^{\mathrm{t}}\right) = 0 \end{cases} \qquad (3)$$

where E is the mathematical expectation.

(iii) disturbances are not correlated, that means $\mathrm{E}\left(\zeta\zeta^{\mathrm{t}}\right)$ is q diagonal matrix.

2 Correlation matrix implied by a recursive path model and its computation.

The correlation matrix implied by the model described by system (1) is the ((p+q)×(p+q)) symetric matrix $\widehat{\mathrm{R}}$ whose elements are the correlations between each pair of variables in the model. This matrix can be defined as :

$$\widehat{\mathrm{R}} = \begin{bmatrix} \mathrm{E}\left(\xi\xi^{\mathrm{t}}\right) & \mathrm{E}\left(\xi\eta^{\mathrm{t}}\right) \\ \mathrm{E}\left(\eta\xi^{\mathrm{t}}\right) & \mathrm{E}\left(\eta\eta^{\mathrm{t}}\right) \end{bmatrix} \qquad (4)$$

2.1 Jöreskog's method

The expression given by Jöreskog to compute the correlation matrix implied by the model is as follows :

$$\widehat{\mathrm{R}}=\widehat{\mathrm{R}}^{\mathrm{JOR}}=\begin{bmatrix} \Phi & \Phi\Gamma^{\mathrm{t}}\left[(\mathrm{I}-\mathrm{B})^{-1}\right]^{\mathrm{t}} \\ (\mathrm{I}-\mathrm{B})^{-1}\Gamma\Phi & (\mathrm{I}-\mathrm{B})^{-1}\left(\Gamma\Phi\Gamma^{\mathrm{t}}+\Psi\right)\left[(\mathrm{I}-\mathrm{B})^{-1}\right]^{\mathrm{t}} \end{bmatrix} \qquad (5)$$

where $\Phi = \mathrm{E}\left(\xi\xi^{\mathrm{t}}\right) = \left[\mathrm{cor}\left(\xi_{\mathrm{j}}, \xi_{\mathrm{i}}\right)\right]_{1\leq \mathrm{j,i}\leq \mathrm{p}}$ is the (p×p) correlation matrix among exogenous variables and $\Psi = \mathrm{E}\left(\zeta\zeta^{\mathrm{t}}\right) = \left[\mathrm{cov}\left(\zeta_{\mathrm{j}}, \zeta_{\mathrm{i}}\right)\right]_{1\leq \mathrm{j,i}\leq \mathrm{q}}$ is the (q×q) covariance matrix among disturbances.

The matrix (5) is derived from (2) that is equivalent to the following equation :

$$(\mathrm{I}-\mathrm{B})\,\eta=\Gamma\xi+\zeta \qquad (6)$$

or equivalently

$$\eta=(\mathrm{I}-\mathrm{B})^{-1}\left(\Gamma\xi+\zeta\right) \qquad (7)$$

The equivalence between (2) and (6) is obvious. Equivalence between (6) and (7) comes from the fact that the lower triangular matrix (I−B) is not singular because its diagonal elements are nonzero (all are equal to 1).

In fact,

$$\mathrm{E}\left(\eta\xi^{\mathrm{t}}\right) = \mathrm{E}\left((\mathrm{I}-\mathrm{B})^{-1}\left(\Gamma\xi+\zeta\right)\xi^{\mathrm{t}}\right) = \mathrm{E}\left((\mathrm{I}-\mathrm{B})^{-1}\Gamma\xi\xi^{\mathrm{t}}+(\mathrm{I}-\mathrm{B})^{-1}\zeta\xi^{\mathrm{t}}\right)$$

thus

$$\mathrm{E}\left(\eta\xi^{\mathrm{t}}\right)=(\mathrm{I}-\mathrm{B})^{-1}\Gamma\mathrm{E}\left(\xi\xi^{\mathrm{t}}\right)+(\mathrm{I}-\mathrm{B})^{-1}\mathrm{E}\left(\zeta\xi^{\mathrm{t}}\right)$$

however, from (3),

$$\mathrm{E}\left(\zeta\xi^{\mathrm{t}}\right) = 0$$

thus

$$\mathrm{E}\left(\eta\xi^{\mathrm{t}}\right)=(\mathrm{I}-\mathrm{B})^{-1}\Gamma\Phi \qquad (8)$$

and by transposition and since Φ is symetric :

$$\mathrm{E}\left(\xi\eta^{\mathrm{t}}\right)=\left[\mathrm{E}\left(\eta\xi^{\mathrm{t}}\right)\right]^{\mathrm{t}}=[(\mathrm{I}-\mathrm{B})^{-1}\Gamma\Phi]^{\mathrm{t}}=\Phi\Gamma^{\mathrm{t}}\left[(\mathrm{I}-\mathrm{B})^{-1}\right]^{\mathrm{t}} \qquad (9)$$

otherwise,

$$\mathrm{E}\left(\eta\eta^{\mathrm{t}}\right)=\mathrm{E}\left(\left[(\mathrm{I}-\mathrm{B})^{-1}\left(\Gamma\xi+\zeta\right)\right]\left[(\mathrm{I}-\mathrm{B})^{-1}\left(\Gamma\xi+\zeta\right)\right]^{\mathrm{t}}\right)=\mathrm{E}\left[(\mathrm{I}-\mathrm{B})^{-1}\left(\Gamma\xi+\zeta\right)\left(\Gamma\xi+\zeta\right)^{\mathrm{t}}\left[(\mathrm{I}-\mathrm{B})^{-1}\right]^{\mathrm{t}}\right]$$

it follows :

$$\mathrm{E}\left(\eta\eta^{\mathrm{t}}\right)=(\mathrm{I}-\mathrm{B})^{-1}\left[\Gamma\mathrm{E}\left(\xi\xi^{\mathrm{t}}\right)\Gamma^{\mathrm{t}}+\Gamma\mathrm{E}\left(\xi\zeta^{\mathrm{t}}\right)+\mathrm{E}\left(\zeta\xi^{\mathrm{t}}\right)\Gamma^{\mathrm{t}}+\mathrm{E}\left(\zeta\zeta^{\mathrm{t}}\right)\right]\left[(\mathrm{I}-\mathrm{B})^{-1}\right]^{\mathrm{t}}$$

since $\mathrm{E}\left(\xi\xi^{\mathrm{t}}\right)=\Phi$, $\mathrm{E}\left(\xi\zeta^{\mathrm{t}}\right)=0$, $\mathrm{E}\left(\zeta\xi^{\mathrm{t}}\right)=0$ and $\mathrm{E}\left(\zeta\zeta^{\mathrm{t}}\right)=\Psi$ it follows,

$$\mathrm{E}\left(\eta\eta^{\mathrm{t}}\right)=(\mathrm{I}-\mathrm{B})^{-1}\left(\Gamma\Phi\Gamma^{\mathrm{t}}+\Psi\right)\left[(\mathrm{I}-\mathrm{B})^{-1}\right]^{\mathrm{t}} \qquad (10)$$

substituting (8), (9) and (10) in (4) gives (5).

2.2 The Finite Iterative Method

This part presents FIM, a new method to compute the matrix $\widehat{\mathrm{R}}$ with a different vision. Indeed, its principle shown in Figure 2 is to build $\widehat{\mathrm{R}}$ iteratively. Starting with the first square block of order p (corresponding to the p exogenous variables $(\xi_1, \xi_2, \ldots, \xi_{\mathrm{p}})$), then the $(\mathrm{p}+1)^{\mathrm{th}}$ row and the $(\mathrm{p}+1)^{\mathrm{th}}$ column (corresponding to the first endogenous variable η_1) and ending with the $(\mathrm{p}+\mathrm{q})^{\mathrm{th}}$ row and $(\mathrm{p}+\mathrm{q})^{\mathrm{th}}$ column (corresponding to the last endogenous variable η_{q}).

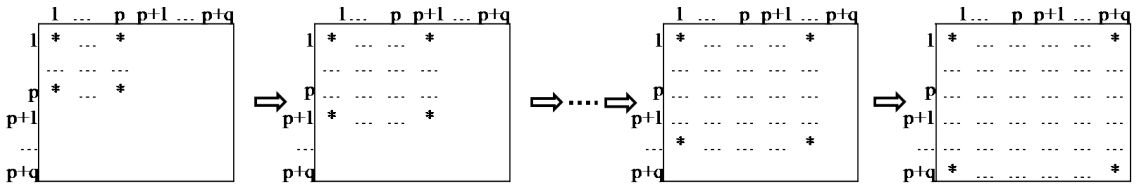

Figure 2: Construction of the implied correlation matrix using the Finite Iterative Method

This building process is possible by using system (1) directly without recourse to (7). This system can be written as follows :

$$\begin{cases} \eta_1=\gamma_{11}\xi_1+\ldots+\gamma_{1p}\xi_p+\zeta_1 \\ \eta_2=\gamma_{21}\xi_1+\ldots+\gamma_{2p}\xi_p+\beta_{21}\eta_1+\zeta_2 \\ \vdots \\ \eta_j=\gamma_{j1}\xi_1+\ldots+\gamma_{j,p}\xi_p+\beta_{j1}\eta_1+\ldots+\beta_{j,(j-1)}\eta_{j-1}+\zeta_j \\ \vdots \\ \eta_q=\gamma_{q1}\xi_1+\ldots+\gamma_{q,p}\xi_p+\beta_{q1}\eta_1+\ldots+\beta_{q,(q-1)}\eta_{q-1}+\zeta_q \end{cases}$$

Thereafter, we denote by A the $(q\times(p+q))$ matrix of ***model parameters*** defined as :

$$A=\begin{bmatrix} \Gamma & B \end{bmatrix}=\begin{bmatrix} \gamma_{11} & \cdots & \gamma_{1p} & 0 & \cdots & \cdots & 0 \\ \gamma_{21} & \cdots & \gamma_{2p} & \beta_{21} & \ddots & \ddots & \vdots \\ \cdots & \cdots & \cdots & \cdots & \ddots & \ddots & \vdots \\ \gamma_{q1} & \cdots & \gamma_{q,p} & \beta_{q1} & \cdots & \beta_{q,(q-1)} & 0 \end{bmatrix}$$

For fixed k between 1 and q, and m between 1 and $(p+q)$, the following notations are used :

$$A_{1:k,1:m}=[a_{i,j}]_{1\le i\le k,1\le j\le m}$$

The Finite iterative method is defined by the following q (finite) iterations (Algorithm 1):

Repeat for $j=1,2,\ldots,q$:

1. $\widehat{R}^{FIM}_{p+j,1:p+j-1}=A_{j,1:p+j-1}\ \widehat{R}^{FIM}_{1:p+j-1,1:p+j-1}$
2. $\widehat{R}^{FIM}_{1:p+j-1,p+j}=\left(\widehat{R}^{FIM}_{p+j,1:p+j-1}\right)^t$
3. $\widehat{R}^{FIM}_{p+j,p+j}=1$

Algorithm 1. Finite Iterative Method

These iterations are initializated by $\widehat{R}^{FIM}_{1:p,1:p}=\Phi$

Theorem. *The matrix* $\widehat{R}$ *can be computed by the algorithm above :* $\widehat{R}=\widehat{R}^{FIM}$

Proof of theorem

Variables $\xi_1, \ldots$and ξ_p are all exogenous, therefore the block $\widehat{R}^{FIM}_{1:p,1:p}$ corresponding to these variables is naturally identical to Φ, the correlation matrix among exogenous variables. That means,

$$\widehat{R}^{FIM}_{1:p,1:p}=\Phi$$

1. The following notations are made :

(i) $\widehat{r}_{\xi_i\xi_{i'}}= cor\,(\xi_i, \xi_{i'}) = E\,(\xi_{i'}\xi_i) \qquad \forall(i, i') \in \{1, p\}$
(ii) $\widehat{r}_{\eta_j\xi_i}= cor\,(\eta_j, \xi_i) = E\,(\eta_j\xi_i) \qquad \forall j\in \{1, q\}$ and $\forall i \in \{1, p\}$
(iii) $\widehat{r}_{\eta_j\eta_{j'}}= cor\,(\eta_j, \eta_{j'}) = E\,(\eta_{j'}\eta_j) \qquad \forall(j, j') \in \{1, q\}$

Since the first equation contains in the right term only the exogenous variables, we separate it from the other equations. This first equation is,

$$\eta_1=\gamma_{11}\xi_1+\ldots+\gamma_{1p}\xi_p+\zeta_1$$

thus, for $i \in \{1, p\}$ the multiplication of this equation on the right side by ξ_i gives,

$$\eta_1\xi_i =\gamma_{11}\ \xi_1\xi_i+\ldots+\gamma_{1,p}\ \xi_p\xi_i+\ \zeta_1\xi_i$$

however, models assumptions in (3) imply that ζ_1 is uncorrelated with all exogenous variables $\xi_1, \ldots$and ξ_p. This implies,

$$E(\zeta_1\xi_i) = 0 \qquad \forall i\in \{1, p\}$$

so, taking the respective mathematical expectation we obtain,

$$\widehat{r}_{\eta_1\xi_i}=\gamma_{1,1}\widehat{r}_{\xi_1\xi_i}+\ldots+\gamma_{1,p}\widehat{r}_{\xi_p\xi_i}$$

or equivalently,

$$\widehat{r}_{\eta_1\xi_i}=A_{1,1:p}\ \widehat{R}^{FIM}_{1:p,i}$$

thus, for i in the set of integers between 1 and p we obtain,

$$\widehat{R}^{FIM}_{p+1,1:p} = \left[\widehat{r}_{\eta_1\xi_1}, \ldots, \widehat{r}_{\eta_1\xi_p}\right] = \left[A_{1,1:p}\ \widehat{R}^{FIM}_{1:p,1}, \ldots,\ A_{1,1:p}\ \widehat{R}^{FIM}_{1:p,p}\right] = A_{1,1:p}\ \widehat{R}^{FIM}_{1:p,1:p}$$

Now we consider equations for the other endogenous variables $\eta_2, \ldots$and η_q.

Let $j \in \{2, q\}$, the structural equation for the j^{th} endogenous variable η_j is :

$$\eta_j=\gamma_{j1}\xi_1+\ldots+\gamma_{j,p}\xi_p+\beta_{j1}\eta_1+\ldots+\beta_{j,(j-1)}\eta_{j-1}+\zeta_j$$

let $\mathrm{i} \in \{1, \mathrm{p}\}$ and $\mathrm{k} \in \{1, \mathrm{j}-1\}$, if we multiply this equation on the right side successively by ξ_i and η_k we obtain,

$$\eta_\mathrm{j}\xi_\mathrm{i} = \gamma_{\mathrm{j}1}\ \xi_1\xi_\mathrm{i} + \ldots + \gamma_{\mathrm{j,p}}\ \xi_\mathrm{p}\xi_\mathrm{i} + \beta_{\mathrm{j}1}\ \eta_1\xi_\mathrm{i} + \ldots + \beta_{\mathrm{j},(\mathrm{j}-1)}\ \eta_{\mathrm{j}-1}\xi_\mathrm{i} + \zeta_\mathrm{j}\xi_\mathrm{i}$$

and

$$\eta_\mathrm{j}\eta_\mathrm{k} = \gamma_{\mathrm{j}1}\ \xi_1\eta_\mathrm{k} + \ldots + \gamma_{\mathrm{j,p}}\ \xi_\mathrm{p}\eta_\mathrm{k} + \beta_{\mathrm{j}1}\ \eta_1\eta_\mathrm{k} + \ldots + \beta_{\mathrm{j},(\mathrm{j}-1)}\ \eta_{\mathrm{j}-1}\eta_\mathrm{k} + \zeta_\mathrm{j}\eta_\mathrm{k}$$

From (3) ζ_j is uncorrelated with all exogenous variables $\xi_1, \ldots$and ξ_p. Moreover, since η_j is explained by $\eta_1, \ldots$and $\eta_{\mathrm{j}-1}$, ζ_j is uncorrelated with all the endogenous variables $\eta_1, \ldots$and $\eta_{\mathrm{j}-1}$. Thus,

$$\begin{cases} \mathrm{E}\left(\zeta_\mathrm{j}\xi_\mathrm{i}\right) = 0 & \forall \mathrm{i} \in \{1, \mathrm{p}\} \\ \mathrm{E}\left(\zeta_\mathrm{j}\eta_\mathrm{k}\right) = 0 & \forall \mathrm{k} \in \{1, \mathrm{j}-1\} \end{cases}$$

so, taking the respective mathematical expectation, we obtain,

$$\widehat{\mathrm{r}}_{\eta_\mathrm{j}\xi_\mathrm{i}} = \gamma_{\mathrm{j},1}\widehat{\mathrm{r}}_{\xi_1\xi_\mathrm{i}} + \ldots + \gamma_{\mathrm{j,p}}\widehat{\mathrm{r}}_{\xi_\mathrm{p}\xi_\mathrm{i}} + \beta_{\mathrm{j},1}\widehat{\mathrm{r}}_{\eta_1\xi_\mathrm{i}} + \ldots + \beta_{\mathrm{j,j}-1}\widehat{\mathrm{r}}_{\eta_{\mathrm{j}-1}\xi_\mathrm{i}}$$

and

$$\widehat{\mathrm{r}}_{\eta_\mathrm{j}\eta_\mathrm{k}} = \gamma_{\mathrm{j},1}\widehat{\mathrm{r}}_{\xi_1\eta_\mathrm{k}} + \ldots + \gamma_{\mathrm{j,p}}\widehat{\mathrm{r}}_{\xi_\mathrm{p}\eta_\mathrm{k}} + \beta_{\mathrm{j},1}\widehat{\mathrm{r}}_{\eta_1\eta_\mathrm{k}} + \ldots + \beta_{\mathrm{j,j}-1}\widehat{\mathrm{r}}_{\eta_{\mathrm{j}-1}\eta_\mathrm{k}}$$

or equivalently,

$$\widehat{\mathrm{r}}_{\eta_\mathrm{j}\xi_\mathrm{i}} = \mathrm{A}_{\mathrm{j},1:\mathrm{p}+\mathrm{j}-1}\ \widehat{\mathrm{R}}^{\mathrm{FIM}}_{1:\mathrm{p}+\mathrm{j}-1,\mathrm{i}}$$

and

$$\widehat{\mathrm{r}}_{\eta_\mathrm{j}\eta_\mathrm{k}} = \mathrm{A}_{\mathrm{j},1:\mathrm{p}+\mathrm{j}-1}\ \widehat{\mathrm{R}}^{\mathrm{FIM}}_{1:\mathrm{p}+\mathrm{j}-1,\mathrm{p}+\mathrm{k}}$$

thus for i in the set of integers between 1 and p, and k in the set of integers between 1 and j-1 :

$$\widehat{\mathrm{R}}^{\mathrm{FIM}}_{\mathrm{p}+\mathrm{j},1:\mathrm{p}} = \left[\widehat{\mathrm{r}}_{\eta_\mathrm{j}\xi_1}, \ldots, \widehat{\mathrm{r}}_{\eta_\mathrm{j}\xi_\mathrm{p}}\right] = \left[\mathrm{A}_{\mathrm{j},1:\mathrm{p}+\mathrm{j}-1}\ \widehat{\mathrm{R}}^{\mathrm{FIM}}_{1:\mathrm{p}+\mathrm{j}-1,1}, \ldots,\ \mathrm{A}_{\mathrm{j},1:\mathrm{p}+\mathrm{j}-1}\ \widehat{\mathrm{R}}^{\mathrm{FIM}}_{1:\mathrm{p}+\mathrm{j}-1,\mathrm{p}}\right]$$

and

$$\widehat{\mathrm{R}}^{\mathrm{FIM}}_{\mathrm{p}+\mathrm{j},\mathrm{p}+1:\mathrm{p}+\mathrm{j}-1} = \left[\widehat{\mathrm{r}}_{\eta_\mathrm{j}\eta_1}, \ldots, \widehat{\mathrm{r}}_{\eta_\mathrm{j}\eta_{\mathrm{j}-1}}\right] = \left[\mathrm{A}_{\mathrm{j},1:\mathrm{p}+\mathrm{j}-1}\ \widehat{\mathrm{R}}^{\mathrm{FIM}}_{1:\mathrm{p}+\mathrm{j}-1,\mathrm{p}+1}, \ldots,\ \mathrm{A}_{\mathrm{j},1:\mathrm{p}+\mathrm{j}-1}\ \widehat{\mathrm{R}}^{\mathrm{FIM}}_{1:\mathrm{p}+\mathrm{j}-1,\mathrm{p}+\mathrm{j}-1}\right]$$

thus

$$\widehat{\mathrm{R}}^{\mathrm{FIM}}_{\mathrm{p}+\mathrm{j},1:\mathrm{p}} = \mathrm{A}_{\mathrm{j},1:\mathrm{p}+\mathrm{j}-1}\ \widehat{\mathrm{R}}^{\mathrm{FIM}}_{1:\mathrm{p}+\mathrm{j}-1,1:\mathrm{p}} \qquad (11)$$

and

$$\widehat{\mathrm{R}}^{\mathrm{FIM}}_{\mathrm{p+j,p+1:p+j-1}}=\mathrm{A}_{\mathrm{j,1:p+j-1}}\ \widehat{\mathrm{R}}^{\mathrm{FIM}}_{\mathrm{1:p+j-1,p+1:p+j-1}} \qquad (12)$$

(11) and (12) give

$$\widehat{\mathrm{R}}^{\mathrm{FIM}}_{\mathrm{p+j,1:p+j-1}}=\mathrm{A}_{\mathrm{j,1:p+j-1}}\ \widehat{\mathrm{R}}^{\mathrm{FIM}}_{\mathrm{1:p+j-1,1:p+j-1}}$$

2. Since $\widehat{\mathrm{R}}^{\mathrm{FIM}}$ is a correlation matrix then it is symmetric. This then implies,

$$\widehat{\mathrm{R}}^{\mathrm{FIM}}_{\mathrm{1:p+j-1,p+j}}=\left(\widehat{\mathrm{R}}^{\mathrm{FIM}}_{\mathrm{p+j,1:p+j-1}}\right)^{\mathrm{t}}$$

3. Since $\widehat{\mathrm{R}}^{\mathrm{FIM}}$ is also a correlation matrix then by definition :

$$\widehat{\mathrm{R}}^{\mathrm{FIM}}_{\mathrm{p+j,p+j}}=1$$

3 Efficiency of FIM and its complementarity with Jöreskog's method.

The aim of this section is to analyse the efficiency of FIM compared with Jöreskog's method, and discuss their complementarity. To do this, the model described in Figure 1 is considered. The three structural equations in this model can be written as system (1) with,

$$\mathrm{B}=\begin{bmatrix} 0 & 0 & 0 \\ \mathrm{c} & 0 & 0 \\ \mathrm{d} & \mathrm{e} & 0 \end{bmatrix}, \Gamma=\begin{bmatrix} \mathrm{a} \\ \mathrm{b} \\ 0 \end{bmatrix}, \Phi=[1].$$

3.1 How to calculate $\widehat{\mathrm{R}}$ by the two methods ?

For Jöreskog's method we start with

$$(\mathrm{I}-\mathrm{B})^{-1}=\begin{bmatrix} 1 & 0 & 0 \\ \mathrm{c} & 1 & 0 \\ \mathrm{d+ce} & \mathrm{e} & 1 \end{bmatrix}$$

thus from (8)

$$\mathrm{E}\left(\eta\xi^{\mathrm{t}}\right)=\begin{bmatrix} 1 & 0 & 0 \\ \mathrm{c} & 1 & 0 \\ \mathrm{d+ce} & \mathrm{e} & 1 \end{bmatrix}\begin{bmatrix} \mathrm{a} \\ \mathrm{b} \\ 0 \end{bmatrix}\times 1=\begin{bmatrix} \mathrm{a} \\ \mathrm{ac+b} \\ \mathrm{ad+ace+be} \end{bmatrix}$$

and from (9)

$$\mathrm{E}\left(\xi\eta^{\mathrm{t}}\right)=(\mathrm{E}\left(\eta\xi^{\mathrm{t}}\right))^{\mathrm{t}}=\left[\begin{array}{ccc} \mathrm{a} & \mathrm{ac+b} & \mathrm{ad+ace+be} \end{array}\right]$$

However the matrix Ψ is diagonal with elements :

$$\left\{\begin{array}{l} \theta_1^2=\mathrm{Var}\left(\zeta_1\right) \\ \theta_2^2=\mathrm{Var}\left(\zeta_2\right) \\ \theta_3^2=\mathrm{Var}\left(\zeta_3\right) \end{array}\right.$$

then

$$\Psi=\left[\begin{array}{ccc} \theta_1^2 & 0 & 0 \\ 0 & \theta_2^2 & 0 \\ 0 & 0 & \theta_3^2 \end{array}\right]$$

thus from (10) :

$$\mathrm{E}\left(\eta\eta^{\mathrm{t}}\right)=\left[\begin{array}{ccc} 1 & 0 & 0 \\ \mathrm{c} & 1 & 0 \\ \mathrm{d+ce} & \mathrm{e} & 1 \end{array}\right]\left(\left[\begin{array}{c} \mathrm{a} \\ \mathrm{b} \\ 0 \end{array}\right]\times 1\left[\begin{array}{ccc} \mathrm{a} & \mathrm{b} & 0 \end{array}\right]+\left[\begin{array}{ccc} \theta_1^2 & 0 & 0 \\ 0 & \theta_2^2 & 0 \\ 0 & 0 & \theta_3^2 \end{array}\right]\right)\left[\begin{array}{ccc} 1 & \mathrm{c} & \mathrm{d+ce} \\ 0 & 1 & \mathrm{e} \\ 0 & 0 & 1 \end{array}\right]$$

thus,

$$\mathrm{E}\left(\eta\eta^{\mathrm{t}}\right)=\left[\begin{array}{ccc} 1 & 0 & 0 \\ \mathrm{c} & 1 & 0 \\ \mathrm{d+ce} & \mathrm{e} & 1 \end{array}\right]\left[\begin{array}{ccc} \mathrm{a}^2+\theta_1^2 & \mathrm{ab} & 0 \\ \mathrm{ab} & \mathrm{b}^2+\theta_2^2 & 0 \\ 0 & 0 & \theta_3^2 \end{array}\right]\left[\begin{array}{ccc} 1 & \mathrm{c} & \mathrm{d+ce} \\ 0 & 1 & \mathrm{e} \\ 0 & 0 & 1 \end{array}\right]$$

and therefore, the correlation matrix implied by the model described by system (1) and provided by Jöreskog's method is written as :

$$\widehat{\mathrm{R}}=\widehat{\mathrm{R}}^{\mathrm{JOR}}=\left[\begin{array}{cccc} 1 & \mathrm{a} & \mathrm{b+ac} & \mathrm{ad+be+ace} \\ \mathrm{a} & \mathrm{a}^2+\theta_1^2 & \mathrm{a}\left(\mathrm{ac+b}\right)+\mathrm{c}\theta_1^2 & \mathrm{a}\left[\mathrm{a}\left(\mathrm{d+ce}\right)+\mathrm{be}\right]+\left(\mathrm{d+ce}\right)\theta_1^2 \\ \mathrm{b+ac} & \mathrm{a}\left(\mathrm{ac+b}\right)+\mathrm{c}\theta_1^2 & \left(\mathrm{ac+b}\right)^2+\mathrm{c}^2\theta_1^2+\theta_2^2 & \left(\mathrm{ac+b}\right)\left[\mathrm{a}\left(\mathrm{d+ce}\right)+\mathrm{be}\right]+\mathrm{c}\left(\mathrm{d+ce}\right)\theta_1^2+\mathrm{c}\theta_2^2 \\ \mathrm{ad+be+ace} & \mathrm{a}\left[\mathrm{a}\left(\mathrm{d+ce}\right)+\mathrm{be}\right]+\left(\mathrm{d+ce}\right)\theta_1^2 & \left(\mathrm{ac+b}\right)\left[\mathrm{a}\left(\mathrm{d+ce}\right)+\mathrm{be}\right]+\mathrm{c}\left(\mathrm{d+ce}\right)\theta_1^2+\mathrm{c}\theta_2^2 & \left[\mathrm{a}\left(\mathrm{d+ce}\right)+\mathrm{be}\right]^2+\left(\mathrm{d+ce}\right)^2\theta_1^2+\mathrm{e}\theta_2^2+\theta_3^2 \end{array}\right]$$

(13)

For the Finite iterative method we consider

$$\mathrm{A}=\left[\begin{array}{cccc} \mathrm{a} & 0 & 0 & 0 \\ \mathrm{b} & \mathrm{c} & 0 & 0 \\ 0 & \mathrm{d} & \mathrm{e} & 0 \end{array}\right]$$

.

The correlation matrix implied by the model $\widehat{\mathrm{R}}^{\mathrm{FIM}}$ provided by FIM is built in three iterations (q=3) and the algorithm 1 is initialisated by

$$\widehat{\mathrm{R}}_{1:1,1:1}^{\mathrm{FIM}}=\ \Phi=[1]$$

First iteration (q=1) : Computation for the first endogenous variable.

$\widehat{\mathrm{R}}^{\mathrm{FIM}}_{2,1:1}=\mathrm{A}_{1,1:1}$ $\widehat{\mathrm{R}}^{\mathrm{FIM}}_{1:1,1:1}=\mathrm{a}\times 1=\mathrm{a}$

$\widehat{\mathrm{R}}^{\mathrm{FIM}}_{1:1,2}=\left(\widehat{\mathrm{R}}^{\mathrm{FIM}}_{2,1:1}\right)^{\mathrm{t}}=\mathrm{a}$

$\widehat{\mathrm{R}}^{\mathrm{FIM}}_{2,2}=1$, thus

$$\widehat{\mathrm{R}}^{\mathrm{FIM}}_{1:2,1:2}=\begin{bmatrix} 1 & \mathrm{a} \\ \mathrm{a} & 1 \end{bmatrix}$$

Second iteration (q=2) : Computation for the second endogenous variable.

$$\widehat{\mathrm{R}}^{\mathrm{FIM}}_{3,1:2}=\mathrm{A}_{2,1:2}\ \widehat{\mathrm{R}}^{\mathrm{FIM}}_{1:2,1:2}=\begin{bmatrix} \mathrm{b} & \mathrm{c} \end{bmatrix}\begin{bmatrix} 1 & \mathrm{a} \\ \mathrm{a} & 1 \end{bmatrix}=\begin{bmatrix} \mathrm{b+ac} & \mathrm{ab+c} \end{bmatrix}$$

$$\widehat{\mathrm{R}}^{\mathrm{FIM}}_{1:2,3}=\left(\widehat{\mathrm{R}}^{\mathrm{FIM}}_{3,1:2}\right)^{\mathrm{t}}=\begin{bmatrix} \mathrm{b+ac} \\ \mathrm{ab+c} \end{bmatrix}$$

$\widehat{\mathrm{R}}^{\mathrm{FIM}}_{3,3}=1$, thus

$$\widehat{\mathrm{R}}^{\mathrm{FIM}}_{1:3,1:3}=\begin{bmatrix} 1 & \mathrm{a} & \mathrm{b+ac} \\ \mathrm{a} & 1 & \mathrm{ab+c} \\ \mathrm{b+ac} & \mathrm{ab+c} & 1 \end{bmatrix}$$

Third iteration (q=3) : Computation for the third endogenous variable.

$$\widehat{\mathrm{R}}^{\mathrm{FIM}}_{4,1:3}=\mathrm{A}_{3,1:3}\ \widehat{\mathrm{R}}^{\mathrm{FIM}}_{1:3,1:3}=\begin{bmatrix} 0 & \mathrm{d} & \mathrm{e} \end{bmatrix}\begin{bmatrix} 1 & \mathrm{a} & \mathrm{b+ac} \\ \mathrm{a} & 1 & \mathrm{ab+c} \\ \mathrm{b+ac} & \mathrm{ab+c} & 1 \end{bmatrix}$$

$$=\begin{bmatrix} \mathrm{ad+be+ace} & \mathrm{d+abe+ce} & \mathrm{abd+cd+e} \end{bmatrix}$$

$$\widehat{\mathrm{R}}^{\mathrm{FIM}}_{1:3,4}=\left(\widehat{\mathrm{R}}^{\mathrm{FIM}}_{4,1:3}\right)^{\mathrm{t}}=\begin{bmatrix} \mathrm{ad+be+ace} \\ \mathrm{d+abe+ce} \\ \mathrm{abd+cd+e} \end{bmatrix}$$

$\widehat{\mathrm{R}}^{\mathrm{FIM}}_{4,4}=1$ thus,

$$\widehat{\mathrm{R}}^{\mathrm{FIM}} = \widehat{\mathrm{R}}^{\mathrm{FIM}}_{1:4,1:4} = \begin{bmatrix} 1 & \mathrm{a} & \mathrm{b+ac} & \mathrm{ad+be+ace} \\ \mathrm{a} & 1 & \mathrm{ab+c} & \mathrm{d+abe+ce} \\ \mathrm{b+ac} & \mathrm{ab+c} & 1 & \mathrm{abd+cd+e} \\ \mathrm{ad+be+ace} & \mathrm{d+abe+ce} & \mathrm{abd+cd+e} & 1 \end{bmatrix} = \widehat{\mathrm{R}} \qquad (14)$$

3.2 Which method is more flexible?

The answer to this question is summarized in Figure 3. This figure lists the matrices needed to compute the implied correlation matrix for a given recursive model.

	Jöreskog's method	FIM
Φ	*	*
Γ	*	*
B	*	*
$(\mathrm{I}-\mathrm{B})^{-1}$	*	
Ψ	*	

Figure 3: Comparison between Jöreskog's method and "FIM".

As clearly shown in (5), Jöreskog's method requires prior knowledge of the inverse matrix $(\mathrm{I}-\mathrm{B})^{-1}$, and provides $\hat{\mathrm{R}}$ as function of the variances of disturbances (see (13)). FIM, instead requires neither the inverse matrix $(\mathrm{I}-\mathrm{B})^{-1}$ nor the matrix Ψ providing directly the matrix $\hat{\mathrm{R}}$ (calculations depends on Φ, Γ and B only, which are given). As a consequence, FIM seems more flexible than Jöreskog's method in computing the correlation matrix implied by the model.

3.3 Is FIM more efficient than Jöreskog's method ?

The answer to this question depends on the matrix of variances of disturbance Ψ. When this matrix can be exhibited from Φ, Γ and B, the efficiency of both methods is identical. Unfortunately, no known method allows computation of the matrix Ψ from Φ, Γ and B. This leads to the conclusion that FIM is more efficient than Jöreskog's method. In practice, however, Jöreskog's method does not provide exactly the matrix $\hat{\mathrm{R}}$, it provides an approximation of it, noted by $\hat{\mathrm{R}}^{\mathrm{App}}$ because an approximation of Ψ is used.

To illustrate this situation, 100 data sets were randomly generated. Four variables are considered for each data set. The model in Figure 1 is considered for each data set, the matrix Φ is taken as the empirical correlation matrix between exogenous variables, and the matrices Γ and B are estimated as solutions of the minimization criterion

$\frac{1}{2}\text{tr}\left[\left(\text{R}-\widehat{\text{R}}\right)\right]^2$ with R the empirical correlation. The lavaan package (Rosseel, 2012) is used to estimate the model for each simulation. Thereby, for each data set, the matrix $\hat{\text{R}}^{\text{App}}$ provided by Jöreskog's method is given as ouput by the lavaan package, and the matrix $\hat{\text{R}}^{\text{FIM}}$ is computed by algorithm 1. Then we measure the distance between these two matrices defined as :

$$\Delta = \frac{1}{2}\text{tr}\left[\hat{\text{R}}^{\text{App}} - \hat{\text{R}}^{\text{FIM}}\right]^2$$

Figure 4 displays the quantity Δ for each simulation. This figure shows that this difference is negligible since it does not exceed 0.004, which affirms that these two matrices are very close. The most important result of these simulations is judging the quality of this approximation.

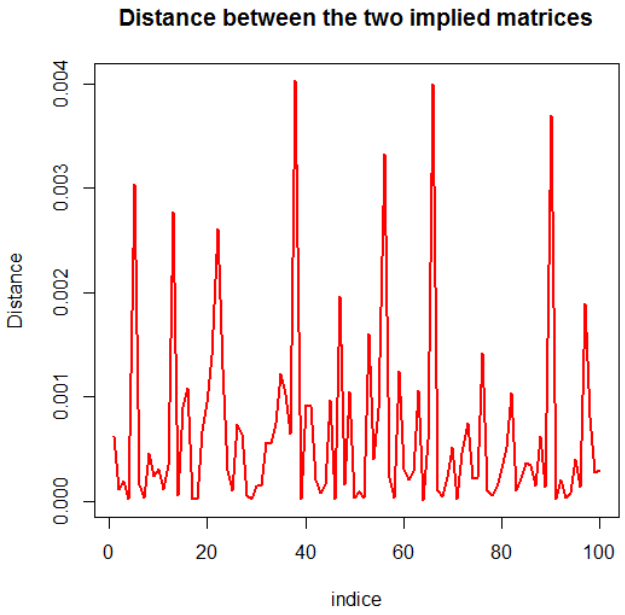

Figure 4: Distance between the two implied matrices $\hat{\text{R}}^{\text{App}}$ and $\hat{\text{R}}^{\text{FIM}}$ for each of 100 simulations.

3.4 Are both methods complementary?

Simultaneous use of both methods has the advantage of determining exactly the matrix of variances of disturbance Ψ as a function of parameters (Φ, Γ and B). This saves the user from worrying about approximations. Indeed, the matrix Ψ can be calculated as follows :

(i) Φ, Γ and B are given,

(ii) compute $\hat{\mathrm{R}}^{\mathrm{FIM}}$,

(iii) compute Ψ as :

$$\Psi{=}(\mathrm{I}{-}\mathrm{B})\widehat{\mathrm{R}}^{\mathrm{FIM}}_{\mathrm{p+1:p+q,p+1:p+q}}(\mathrm{I}{-}\mathrm{B})^{\mathrm{t}}{-}\,\Gamma\Phi\Gamma^{\mathrm{t}} \qquad (15)$$

Indeed, from

$$\widehat{\mathrm{R}}^{\mathrm{JOR}}{=}\widehat{\mathrm{R}}^{\mathrm{FIM}}$$

and by identifying the blocks corresponding to the endogenous variables,

$$(\mathrm{I}{-}\mathrm{B})^{-1}\left(\Gamma\Phi\Gamma^{\mathrm{t}}{+}\Psi\right)\left[(\mathrm{I}{-}\mathrm{B})^{-1}\right]^{\mathrm{t}}{=}\widehat{\mathrm{R}}^{\mathrm{FIM}}_{\mathrm{p+1:p+q,p+1:p+q}}$$

And by applying (15) to the model of Figure 1, the matrix Ψ is as follows :

$$\Psi = \begin{bmatrix} 1 & 0 & 0 \\ -\mathrm{c} & 1 & 0 \\ -\mathrm{d} & -\mathrm{e} & 1 \end{bmatrix} \begin{bmatrix} 1 & \mathrm{ab}+\mathrm{c} & \mathrm{d}+\mathrm{abe}+\mathrm{ce} \\ \mathrm{ab}+\mathrm{c} & 1 & \mathrm{abd}+\mathrm{cd}+\mathrm{e} \\ \mathrm{d}+\mathrm{abe}+\mathrm{ce} & \mathrm{abd}+\mathrm{cd}+\mathrm{e} & 1 \end{bmatrix} \begin{bmatrix} 1 & -\mathrm{c} & -\mathrm{d} \\ 0 & 1 & -\mathrm{e} \\ 0 & 0 & 1 \end{bmatrix}$$

$$- \begin{bmatrix} \mathrm{a} \\ \mathrm{b} \\ 0 \end{bmatrix} \times 1 \times \begin{bmatrix} \mathrm{a} & \mathrm{b} & 0 \end{bmatrix}$$

thus,

$$\Psi = \begin{bmatrix} 1-\mathrm{a}^2 & 0 & 0 \\ 0 & 1-\mathrm{b}^2-\mathrm{c}^2-2\mathrm{abc} & 0 \\ 0 & 0 & 1-\mathrm{d}^2-\mathrm{e}^2-2\mathrm{abde}-2\mathrm{cde} \end{bmatrix}$$

To conclude, FIM does not determine the variances of disturbances, and Jöreskog's method does not allow the computation of $\hat{\mathrm{R}}$. The simultaneous use of the two methods allows calculation of $\hat{\mathrm{R}}$ and Ψ.

4 Conclusion and perspectives.

The computation of the correlation matrix implied by a recursive path model is a crucial step in the global path analysis process. This paper proposes a new method called Finite Iterative Method (FIM) which will enrich and expand strategies for computing this matrix. On a conceptual level, FIM has two advantages compared to the well-known method of Jöreskog. First, it does not need matrix inversion and second, knowledge of variances of disturbances is not necessary. On a practical level, if we get good approximations of variances of disturbances, this new method is equivalent to Jöreskog's method .

The question that still remains to be answered and always open is to propose similar strategies of computation for nonrecursive models, and for models with latent variables.

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