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Reliability estimation of k-unit series system based on progressively censored data

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In this article, we consider a k-unit series system with component lifetime distribution to be a member of the scale family of distributions. We discuss estimation of the scale parameter and estimation of reliability function of the family based on progressively Type-II censored sample. The maximum likelihood estimator (MLE) of the scale parameter is derived using Expectation-Maximization (EM) algorithm and is used to estimate reliability function. Confidence intervals are constructed using asymptotic distribution of MLE. β -expectation tolerance interval for lifetime of the system is obtained. We consider half-logistic distribution as a member of the scale family and study performance of the MLE, reliability estimate and confidence interval using simulation experiments. Illustration through real data example is provided.

keywords: Progressively Type-II censoring, EM algorithm, MLE, confidence interval, coverage probability, reliability, β-expectation tolerance interval, half-logistic distribution.

1 Introduction

In industrial phenomenon series systems are widely used. Electric, automobile as well as in chemical industry various units are connected in series. Here system is working if all

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units in system are working. If any one unit is failed then system fails. Thus, system life is smaller than unit life. Life testing under series system is more costly, because failure of one unit reflects in system failure. Therefore, we use censoring criteria, in that; we remove some working systems without observing its failure time. The unobserved failure time data are called censored data.

Broadly censoring is classified into two types; Type-I and Type-II censoring. Type-I censoring depends on time. In this type, an experiment continues up to a pre-determined time T. Units having failure time after time T are not observed. Here, failure time will be known exactly only if it is less than T. For example, if n units are placed on test, but decision is made to terminate the test at time T, then failure times will be known exactly only for those units that fail before time T. In Type-I censoring, the number of exact failure times observed is random.

Type-II censoring scheme is often used in life testing experiment. In this scheme only m units in a random sample of size $n(m < n)$ are observed. Progressive Type-II censoring is a generalization of Type-II censoring. In progressive censoring scheme, the number m and R_1, R_2,...., R_m are fixed prior to the test and $\sum_{i=1}^{m} R_i = n - m$. At the first failure R_1 units are randomly removed from remaining $n - 1$ units. At the second failure, R_2 units are randomly removed from remaining $n - 2 - R_1$ units and so on. At the m^{th} failure all remaining R_m units are removed. Here, we observe failure time of m units and remaining $n - m$ units are removed at different stages of experiment. In conventional Type-II censoring scheme $R_1 = R_2 = ... = R_{m-1} = 0$ and $R_m = n - m$. In this article, the progressive Type-II censoring scheme is considered.

Many authors studied progressive Type-II censoring scheme for various lifetime distributions. Cohen (1963) introduced progressive Type-II censoring. Mann (1969) and Mann (1971) considered Weibull distribution with progressive censoring. Balakrishnan and Asgharzadeh (2005), Balakrishnan et al. (2003) and Balakrishnan et al. (2004) discussed inference for half-logistic, Gaussian and extreme value distribution under progressive Type-II censoring scheme respectively. Ng (2005) studied parameter estimation for modified Weibull distribution under progressive Type-II censoring. Balakrishnan and Aggarwala (2000) gave details about progressive censoring. Balakrishnan (2007) studied various distributions and inferential methods for progressively censored data. Pradhan (2007) considered point and interval estimation of a k-unit parallel system based on progressive Type-II censoring scheme with exponential distribution as the component life distribution.

Kim and Han (2010) discussed half-logistic distribution for Type-II progressively censored samples. Iliopoulos and Balakrishnan (2011) studied likelihood inference for Laplace distribution based on progressively Type-II censored samples. Asgharzadeh and Valiollahi (2011) considered estimation of the scale parameter of the Lomax distribution under progressive censoring scheme. Krishna and Malik (2012), Krishna and Kumar (2011) and Krishna and Kumar (2013) studied reliability estimation in Maxwell, Lindley and generalized inverted exponential distribution with progressively Type-II censored data. Recently, Potdar and Shirke (2014) discussed inference for the scale parameter of lifetime distribution of k-unit parallel system based on progressively Type-II censored data. Potdar and Shirke (2012) studied inference for the distribution of a k-unit parallel system with exponential distribution as the component life distribution based on Type-II progressively censored sample. Potdar and Shirke (2013a) discussed inference for the parameters of generalized inverted family of distributions. Potdar and Shirke (2013b) studied reliability estimation for the distribution of a k-unit parallel system when Rayleigh distribution as component lifetime distribution.

Dempster et al. (1977) introduced expectation-maximization (EM) algorithm. They presented maximum likelihood estimation for incomplete data. McLachlan and Krishnan (2007) gave more details about EM algorithm. Little and Rubin (2002) have discussed EM algorithm for exponential family of distributions. Pradhan and Kundu (2009) used EM algorithm to estimate parameters of generalized exponential distribution under progressive Type-II censoring scheme. Ng et al. (2002) used EM algorithm to estimate parameters of lognormal and Weibull distributions under Type-II censoring scheme. In this article, we use EM algorithm for estimation of the parameters of a k-unit series system based on progressive Type-II censoring scheme when unit lifetime distribution belongs to the scale family. Parameter estimation is based on the lifetimes of the system. We assume that n units are put on test and failure times of $\sum_{i=1}^{m} R_i = n - m.$ units are censored. Failure times of these censored units are unknown. We consider this data as missing and use EM algorithm to compute MLE. We use idea of missing information principle of Louis (1982). Asymptotic normal distribution of MLE is used to construct confidence interval for the scale parameter. We also discuss tolerance interval for the lifetime of the system, on the lines of Kumbhar and Shirke (2004).

The present work is different than the work reported by Pradhan (2007) in many aspects. The first thing is that we consider scale family of distributions and exponential distribution considered by Pradhan (2007) is a member of the family. Further, we obtain MLE using EM algorithm instead of using Newton-Raphson method. We use Newton-Raphson method within EM algorithm. Pradhan (2007) has considered only parameter estimation, while we consider inference of parameter as well as reliability function. We use missing information principle to compute Fisher information. We illustrate use of the results developed with half-logistic distribution, which is a member of scale family. Number of schemes that we consider are 30, which include schemes with small sample sizes.

In Section 2, we introduce the model and obtain MLE for the scale parameter and reliability function. We also provide an expression for Fisher information. Asymptotic confidence interval for the scale parameter based on MLE, log-MLE and confidence interval for the reliability function is discussed in the same section. Section 3 provides β-expectation tolerance interval for the lifetime of a k-unit series system based on progressively censored data. In Section 4, we consider the half-logistic distribution as a member of the scale family and discuss MLE, reliability function, confidence intervals and tolerance intervals. Performance of the MLE and confidence intervals of scale parameter and reliability function of half-logistic distribution is investigated using simulations. Results of simulation study have been reported in Section 5. Real data application is discussed in Section 6. Conclusions are presented in Section 7.

2 Model and Estimation of the Scale Parameter

Let $\mathbb{G}_\lambda$ be a scale family of lifetime distributions where λ is the parameter of the interest. Consider a k-unit series system with independent and identically distributed units having lifetimes $X_1, X_2, ..., X_k$ of k units. That is, X_i is the lifetime of the i^{th} unit having cumulative distribution function (cdf) $G\left(\frac{x_i}{\lambda}\right)$. The lifetime of the system is $X = Min.(X_1, X_2, ..., X_k)$. The cdf of X is

$$F(x;\lambda) = 1 - \left[1 - G\left(\frac{x}{\lambda}\right)\right]^k \qquad x \geq 0,\ \lambda > 0.$$

The probability density function (pdf) of X is

$$f(x;\lambda) = \frac{k}{\lambda} g\left(\frac{x}{\lambda}\right)\left[1 - G\left(\frac{x}{\lambda}\right)\right]^{k-1} \qquad x \geq 0,\ \lambda > 0.$$

where $g(.)$ is the pdf of X_i when $\lambda = 1$.

2.1 Maximum Likelihood Estimation

Suppose n k-unit series systems are under test and we observe failure times of m systems under progressive type-II censoring. Let $(R_1, R_2, ..., R_m)$ be a progressive censoring scheme.

The likelihood function for the observed data is

$$L(\lambda|\underline{x}) = C\prod_{i=1}^{m} f(x_{(i)};\lambda)\left[1 - F(x_{(i)};\lambda)\right]^{R_i},$$

$$\text{where } C = n\prod_{j=1}^{m-1}\left(n - j - \sum_{i=1}^{j} R_i\right).$$

$$L(\lambda|\underline{x}) = C\prod_{i=1}^{m} \frac{k}{\lambda} g\left(\frac{x_{(i)}}{\lambda}\right)\left[1 - G\left(\frac{x_{(i)}}{\lambda}\right)\right]^{k-1}\left[1 - G\left(\frac{x_{(i)}}{\lambda}\right)\right]^{kR_i},$$

Suppose $x_1, x_2, ..., x_m$ is the observed data and $z_1, z_2, ..., z_m$ is the censored data. We note that z_i is a vector with R_i elements, which is not observable for $i = 1, 2, ..., m$. The censored data $Z = (z_1, z_2, ..., z_m)$ can be considered as missing data.
$X = (x_1, x_2, ..., x_m)$ is observed data. $W = (X, Z)$ is the complete data set. Then complete log-likelihood function is

$$L_c = nlog(k) - nlog(\lambda) + \sum_{i=1}^{m} log\left[g\left(\frac{x_i}{\lambda}\right)\right] + (k-1)\sum_{i=1}^{m} log\left[1 - G\left(\frac{x_i}{\lambda}\right)\right]$$
$$+\sum_{i=1}^{m}\sum_{j=1}^{R_i} log\left[g\left(\frac{z_{ij}}{\lambda}\right)\right] + (k-1)\sum_{i=1}^{m}\sum_{j=1}^{R_i} log\left[1 - G\left(\frac{z_{ij}}{\lambda}\right)\right]. \qquad (1)$$

In order to obtain MLE of λ, we use EM algorithm due to Dempster et al. (1977). For the E step in EM algorithm we take Expectation of Z_{ij}. The derivative of L_c with respect to λ is taken for the M step, where

$$\frac{dL_c}{d\lambda} = -\frac{n}{\lambda} - \frac{1}{\lambda^2}\sum_{i=1}^{m}\frac{x_i g'\left(\frac{x_i}{\lambda}\right)}{g\left(\frac{x_i}{\lambda}\right)} + \frac{(k-1)}{\lambda^2}\sum_{i=1}^{m}\frac{x_i G'\left(\frac{x_i}{\lambda}\right)}{1-G\left(\frac{x_i}{\lambda}\right)}$$

$$-\frac{1}{\lambda^2}\sum_{i=1}^{m}R_i a(x_i,k,\lambda^0) + \frac{(k-1)}{\lambda^2}\sum_{i=1}^{m}R_i b(x_i,k,\lambda^0). \tag{2}$$

$$\text{where } a(x_i,k,\lambda) = E\left[\left.\frac{Z_{ij}g'\left(\frac{Z_{ij}}{\lambda}\right)}{g\left(\frac{Z_{ij}}{\lambda}\right)}\right| Z_{ij} > x_i\right] = \int_{x_i}^{\infty}\frac{zg'\left(\frac{z}{\lambda}\right)}{g\left(\frac{z}{\lambda}\right)}\,\frac{f(z;\lambda)}{1-F(x_i;\lambda)}dz,$$

$$\text{and } b(x_i,k,\lambda) = E\left[\left.\frac{Z_{ij}G'\left(\frac{Z_{ij}}{\lambda}\right)}{1-G\left(\frac{Z_{ij}}{\lambda}\right)}\right| Z_{ij} > x_i\right] = \int_{x_i}^{\infty}\frac{zG'\left(\frac{z}{\lambda}\right)}{1-G\left(\frac{z}{\lambda}\right)}\,\frac{f(z;\lambda)}{1-F(x_i;\lambda)}dz.$$

We have to solve equation $\frac{dL_c}{d\lambda} = 0$ to obtain λ^1 as the solution. But this equation does not have solution in the closed form. Therefore we use Newton-Raphson method and compute λ^1. By using λ^1, we compute $a(x_i,k,\lambda^1)$ and $b(x_i,k,\lambda^1)$. This ends M-step. We continue this procedure until convergence takes place.

In Newton-Raphson method, we have to choose initial value of λ. We use least square estimate. Ng (2005) discussed estimation of model parameters of modified Weibull distribution based on progressively Type-II censored data where the empirical distribution function is computed as (see Meeker and Escobar (1998))

$$\hat{F}(x_i) = 1 - \prod_{j=1}^{i}(1-\hat{p}_j),$$

with

$$\hat{p}_j = \frac{1}{n - \sum_{k=2}^{j} R_{k-1} - j + 1} \qquad \text{for } j = 1,2,.....,m.$$

The estimate of the parameter can be obtained by using least square fit of simple linear regression.

$$y_i = \beta x_i \quad \text{with} \quad \beta = \frac{1}{\lambda}$$

$$y_i = G^{-1}\left[1 - \frac{\left[1-\hat{F}(x_{i-1})\right]^{1/k} + \left[1-\hat{F}(x_i)\right]^{1/k}}{2}\right] \quad \text{for } i = 1,2,.....,m.$$

$$\hat{F}(x_0) = 0,$$

The least square estimates of λ is given by

$$\hat{\lambda}_0 = \frac{\sum_{i=1}^{m} x_i^2}{\sum_{i=1}^{m} x_i \, y_i},$$

We use $\hat{\lambda}_0$ as an initial value of λ to obtain the MLE $\hat{\lambda}_n$ using Newton-Raphson method. Reliability function at time t is

$$R(t) = \left[1 - G\left(\frac{t}{\lambda}\right)\right]^k \qquad t \geq 0, \ \lambda > 0.$$

The Maximum likelihood estimator of $R(t)$ is

$$\hat{R}_n(t) = \left[1 - G\left(\frac{t}{\hat{\lambda}_n}\right)\right]^k \qquad t \geq 0.$$

2.2 Fisher Information

We compute observed Fisher information using the idea of missing information principle of Louis (1982).

Thus, observed information = complete information - missing information.

$$I_x(\lambda) = I_w(\lambda) - I_{w|x}(\lambda),$$

where the complete information = $I_w(\lambda) = -E\left[\frac{d^2L}{d\lambda^2}\right]$ and L is the log-likelihood function based on all n observations. We obtain $I_w(\lambda)$ and $I_{w|x}(\lambda)$ in the following.

Now,

$$L = nlog(k) - nlog(\lambda) + \sum_{i=1}^{n} log\left[g\left(\frac{x_i}{\lambda}\right)\right] + (k-1)\sum_{i=1}^{n} log\left[1 - G\left(\frac{x_i}{\lambda}\right)\right]. \qquad (3)$$

and

$$\frac{dL}{d\lambda} = -\frac{n}{\lambda} - \frac{1}{\lambda^2}\sum_{i=1}^{n} \frac{x_i g'\left(\frac{x_i}{\lambda}\right)}{g\left(\frac{x_i}{\lambda}\right)} + \frac{(k-1)}{\lambda^2}\sum_{i=1}^{n} \frac{x_i G'\left(\frac{x_i}{\lambda}\right)}{1 - G\left(\frac{x_i}{\lambda}\right)}.$$

$$\frac{d^2L}{d\lambda^2} = \frac{n}{\lambda^2} + \frac{1}{\lambda^4}\sum_{i=1}^{n} \frac{x_i^2 g\left(\frac{x_i}{\lambda}\right) g''\left(\frac{x_i}{\lambda}\right) - x_i^2\left[g'\left(\frac{x_i}{\lambda}\right)\right]^2 + 2\lambda x_i g\left(\frac{x_i}{\lambda}\right) g'\left(\frac{x_i}{\lambda}\right)}{\left[g\left(\frac{x_i}{\lambda}\right)\right]^2}$$

$$-\frac{(k-1)}{\lambda^4}\sum_{i=1}^{n} \frac{x_i^2\left[1 - G\left(\frac{x_i}{\lambda}\right)\right] G''\left(\frac{x_i}{\lambda}\right) + x_i^2\left[G'\left(\frac{x_i}{\lambda}\right)\right]^2 + 2\lambda x_i\left[1 - G\left(\frac{x_i}{\lambda}\right)\right] G'\left(\frac{x_i}{\lambda}\right)}{\left[1 - G\left(\frac{x_i}{\lambda}\right)\right]^2}.$$

Complete information is given by

$$I_w(\lambda) = -\frac{n}{\lambda^2} - \frac{1}{\lambda^4}\sum_{i=1}^{n} E\left[\frac{X_i^2 g\left(\frac{X_i}{\lambda}\right) g''\left(\frac{X_i}{\lambda}\right) - X_i^2\left[g'\left(\frac{X_i}{\lambda}\right)\right]^2 + 2\lambda X_i g\left(\frac{X_i}{\lambda}\right) g'\left(\frac{X_i}{\lambda}\right)}{\left[g\left(\frac{X_i}{\lambda}\right)\right]^2}\right]$$

$$+\frac{(k-1)}{\lambda^4}\sum_{i=1}^{n} E\left[\frac{X_i^2\left[1-G\left(\frac{X_i}{\lambda}\right)\right] G''\left(\frac{X_i}{\lambda}\right) + X_i^2\left[G'\left(\frac{X_i}{\lambda}\right)\right]^2 + 2\lambda X_i\left[1-G\left(\frac{X_i}{\lambda}\right)\right] G'\left(\frac{X_i}{\lambda}\right)}{\left[1-G\left(\frac{X_i}{\lambda}\right)\right]^2}\right]. \tag{4}$$

Missing information is given by

$$I_{w|x}(\lambda) = \sum_{i=1}^{m} R_i I_{w|x}^{(i)}(\lambda) = -\sum_{i=1}^{m}\sum_{j=1}^{R_i} E_{Z|X}\left[\frac{d^2 log\left(f(Z_{ij}|x_i,\lambda)\right)}{d\lambda^2}\right]$$

Consider

$$f_{Z|X}(z_{ij}|x_i,\lambda) = \frac{f(z_{ij};\lambda)}{1-F(x_i,\lambda)} = \frac{\frac{k}{\lambda} g\left(\frac{z_{ij}}{\lambda}\right)\left[1-G\left(\frac{z_{ij}}{\lambda}\right)\right]^{k-1}}{\left[1-G\left(\frac{x_i}{\lambda}\right)\right]^k}.$$

Therefore,

$$log(f) = log(k) - log(\lambda) + log\left[g\left(\frac{z_{ij}}{\lambda}\right)\right] + (k-1)log\left[1-G\left(\frac{z_{ij}}{\lambda}\right)\right] - k log\left[1-G\left(\frac{x_i}{\lambda}\right)\right].$$

$$\frac{dlogf}{d\lambda} = -\frac{1}{\lambda} - \frac{z_{ij} g'\left(\frac{z_{ij}}{\lambda}\right)}{\lambda^2 g\left(\frac{z_{ij}}{\lambda}\right)} + \frac{(k-1) z_{ij} G'\left(\frac{z_{ij}}{\lambda}\right)}{\lambda^2\left[1-G\left(\frac{z_{ij}}{\lambda}\right)\right]} - \frac{k x_i G'\left(\frac{x_i}{\lambda}\right)}{\lambda^2\left[1-G\left(\frac{x_i}{\lambda}\right)\right]}.$$

and

$$\frac{d^2logf}{d\lambda^2} = \frac{1}{\lambda^2} + \frac{z_{ij}^2 g\left(\frac{z_{ij}}{\lambda}\right) g''\left(\frac{z_{ij}}{\lambda}\right) - z_{ij}^2\left[g'\left(\frac{z_{ij}}{\lambda}\right)\right]^2 + 2\lambda z_{ij} g\left(\frac{z_{ij}}{\lambda}\right) g'\left(\frac{z_{ij}}{\lambda}\right)}{\lambda^4\left[g\left(\frac{z_{ij}}{\lambda}\right)\right]^2}$$

$$-\frac{(k-1)\left\{z_{ij}^2\left[1-G\left(\frac{z_{ij}}{\lambda}\right)\right] G''\left(\frac{z_{ij}}{\lambda}\right) + z_{ij}^2\left[G'\left(\frac{z_{ij}}{\lambda}\right)\right]^2 + 2\lambda z_{ij}\left[1-G\left(\frac{z_{ij}}{\lambda}\right)\right] G'\left(\frac{z_{ij}}{\lambda}\right)\right\}}{\lambda^4\left[1-G\left(\frac{z_{ij}}{\lambda}\right)\right]^2}$$

$$+\frac{k\left\{x_i^2\left[1-G\left(\frac{x_i}{\lambda}\right)\right] G''\left(\frac{x_i}{\lambda}\right) + x_i^2\left[G'\left(\frac{x_i}{\lambda}\right)\right]^2 + 2\lambda x_i G'\left(\frac{x_i}{\lambda}\right)\left[1-G\left(\frac{x_i}{\lambda}\right)\right]\right\}}{\lambda^4\left[1-G\left(\frac{x_i}{\lambda}\right)\right]^2}.$$

Hence, missing information is

$$I_{w|x}(\lambda) = \sum_{i=1}^{m} R_i I_{w|x}^{(i)}(\lambda) = -\sum_{i=1}^{m}\sum_{j=1}^{R_i} E_{Z|X}\left[\frac{d^2 log\left(f(Z_{ij}|x_i,\lambda)\right)}{d\lambda^2}\right]$$

$$
\begin{aligned}
&= -\frac{n-m}{\lambda^2} \\
&-\frac{1}{\lambda^4}\sum_{i=1}^{m}\sum_{j=1}^{R_i} E\left[\frac{Z_{ij}^2 g\left(\frac{Z_{ij}}{\lambda}\right) g''\left(\frac{Z_{ij}}{\lambda}\right) - Z_{ij}^2\left[g'\left(\frac{Z_{ij}}{\lambda}\right)\right]^2 + 2\lambda Z_{ij} g\left(\frac{Z_{ij}}{\lambda}\right) g'\left(\frac{Z_{ij}}{\lambda}\right)}{\left[g\left(\frac{Z_{ij}}{\lambda}\right)\right]^2}\right] \\
&-\frac{(k-1)}{\lambda^4}\sum_{i=1}^{m}\sum_{j=1}^{R_i} E\left[\frac{Z_{ij}^2\left[1-G\left(\frac{Z_{ij}}{\lambda}\right)\right] G''\left(\frac{Z_{ij}}{\lambda}\right) + Z_{ij}^2\left[G'\left(\frac{Z_{ij}}{\lambda}\right)\right]^2}{\left[1-G\left(\frac{Z_{ij}}{\lambda}\right)\right]^2}\right] \\
&-\frac{2(k-1)}{\lambda^3}\sum_{i=1}^{m}\sum_{j=1}^{R_i} E\left[\frac{Z_{ij}\left[1-G\left(\frac{Z_{ij}}{\lambda}\right)\right] G'\left(\frac{Z_{ij}}{\lambda}\right)}{\left[1-G\left(\frac{Z_{ij}}{\lambda}\right)\right]^2}\right] \\
&+\frac{k}{\lambda^4}\sum_{i=1}^{m} R_i\left[\frac{x_i^2\left[1-G\left(\frac{x_i}{\lambda}\right)\right] G''\left(\frac{x_i}{\lambda}\right) + x_i^2\left[G'\left(\frac{x_i}{\lambda}\right)\right]^2 + 2\lambda x_i G'\left(\frac{x_i}{\lambda}\right)\left[1-G\left(\frac{x_i}{\lambda}\right)\right]}{\left[1-G\left(\frac{x_i}{\lambda}\right)\right]^2}\right]. \qquad (5)
\end{aligned}
$$

Using expressions in equations (4) and (5) we obtain observed Fisher information.

2.3 Confidence Intervals

By using asymptotic normal distribution of MLE $\hat{\lambda}_n$, we construct confidence interval for λ. Let $\hat{\sigma}^2(\hat{\lambda_n}) = \frac{1}{I(\hat{\lambda_n})}$ is the estimated variance of $\hat{\lambda}_n$. Therefore, $100(1-\alpha)\%$ asymptotic confidence interval for λ is given by

$$\left(\hat{\lambda}_n - \tau_{\alpha/2}\sqrt{\hat{\sigma}^2(\hat{\lambda}_n)}, \quad \hat{\lambda}_n + \tau_{\alpha/2}\sqrt{\hat{\sigma}^2(\hat{\lambda}_n)}\right), \qquad (6)$$

where $\tau_{\alpha/2}$ is the upper $100(\alpha/2)^{th}$ percentile of standard normal distribution.

Meeker and Escobar (1998) reported that the asymptotic confidence interval for λ can be computed using $log(\hat{\lambda}_n)$. An approximate $100(1-\alpha)\%$ confidence interval for $log(\lambda)$ is given by

$$\left(log(\hat{\lambda}_n) - \tau_{\alpha/2}\sqrt{\hat{\sigma}^2(log(\hat{\lambda}_n))}, \quad log(\hat{\lambda}_n) + \tau_{\alpha/2}\sqrt{\hat{\sigma}^2(log(\hat{\lambda}_n))}\right),$$

where $\hat{\sigma}^2(log(\hat{\lambda_n}))$ is the estimated variance of $log(\hat{\lambda}_n)$ which is approximated by $\hat{\sigma}^2(log(\hat{\lambda_n})) \approx \frac{\hat{\sigma}^2(\hat{\lambda_n})}{\hat{\lambda_n}^2}$. Hence, an approximate $100(1-\alpha)\%$ confidence interval for λ is given by

$$\left(\hat{\lambda}_n\, e^{\left(-\frac{\tau_{\alpha/2}\sqrt{\hat{\sigma}^2(\hat{\lambda}_n)}}{\hat{\lambda}_n}\right)}, \quad \hat{\lambda_n}\, e^{\left(\frac{\tau_{\alpha/2}\sqrt{\hat{\sigma}^2(\hat{\lambda}_n)}}{\hat{\lambda}_n}\right)}\right). \qquad (7)$$

Let $\hat{R}_n$ is the MLE of reliability function $R(t)$ and $\sigma^2(\hat{R}_n)$ is the variance of $\hat{R}_n$, where

$$\hat{\sigma}^2(\hat{R}_n) \approx \frac{k^2t^2}{\hat{\lambda}_n^4}\left[1-G\left(\frac{t}{\hat{\lambda}_n}\right)\right]^{2(k-1)}\left[G'\left(\frac{t}{\hat{\lambda}_n}\right)\right]^2\hat{\sigma}^2(\hat{\lambda}_n)$$

Therefore, $100(1-\alpha)\%$ asymptotic confidence interval for $R(t)$ is given by

$$\left(\hat{R}_n-\tau_{\alpha/2}\sqrt{\hat{\sigma}^2(\hat{R}_n)},\quad \hat{R}_n+\tau_{\alpha/2}\sqrt{\hat{\sigma}^2(\hat{R}_n)}\right), \tag{8}$$

3 Tolerance Intervals

Kumbhar and Shirke (2004) derived the expression for β-expectation tolerance interval for the lifetime distribution of a k-unit parallel system with component life as exponential distribution. They investigated the performance of the tolerance interval based on complete data. We study the performance of the tolerance interval for the lifetime distribution of a k-unit series system based on progressively Type-II censored data for the scale family of distributions. Let $l_\beta(\lambda)$ be the lower quantile of order β of the cdf $F(x;\lambda)$. Then, we have

$$l_\beta(\lambda) = \lambda G^{-1}\left[1-(1-\beta)^{1/k}\right].$$

Thus, an upper β-expectation tolerance interval for $F(x;\lambda)$ is obtained by

$$I_\beta = (0, l_\beta(\lambda)).$$

The maximum likelihood estimator of $l_\beta(\lambda)$ is given by

$$l_\beta(\hat{\lambda}_n) = \hat{\lambda}_n\, G^{-1}\left[1-(1-\beta)^{1/k}\right],$$

yielding an approximate β- expectation tolerance interval as

$$\hat{I}_\beta = \left(0,\ l_\beta(\hat{\lambda}_n)\right).$$

The expectation of $\hat{I}_\beta$ can be obtained approximately using the approach suggested by Atwood (1984) and given as,

$$E\left[F(I_\beta(\hat{\lambda}_n);\lambda)\right] \approx \beta - 0.5\, F_{02}\, \sigma^2(\hat{\lambda}_n) + \frac{F_{01}\,\sigma^2(\hat{\lambda}_n)\, F_{11}}{F_{10}}, \tag{9}$$

$$\text{where } F_{10} = \frac{dF}{dx},\quad F_{01} = \frac{dF}{d\lambda},\quad F_{11} = \frac{d^2F}{dxd\lambda},\quad F_{02} = \frac{d^2F}{d\lambda^2},$$

$$F_{10} = \frac{k}{\lambda}\left[1-G\left(\frac{x}{\lambda}\right)\right]^{k-1} g\left(\frac{x}{\lambda}\right),\quad F_{01} = -\frac{kx}{\lambda^2}\left[1-G\left(\frac{x}{\lambda}\right)\right]^{k-1} G'\left(\frac{x}{\lambda}\right),$$

$$F_{11} = -\frac{k}{\lambda^3}\left[1-G\left(\frac{x}{\lambda}\right)\right]^{k-2} \times$$

$$\left\{x\left[1-G\left(\frac{x}{\lambda}\right)\right]g'\left(\frac{x}{\lambda}\right)+x(k-1)G'\left(\frac{x}{\lambda}\right)g\left(\frac{x}{\lambda}\right)+\lambda\left[1-G\left(\frac{x}{\lambda}\right)\right]g\left(\frac{x}{\lambda}\right)\right\},$$

$$F_{02}=\frac{kx}{\lambda^4}\left[1-G\left(\frac{x}{\lambda}\right)\right]^{k-2}\times$$
$$\left\{x\left[1-G\left(\frac{x}{\lambda}\right)\right]G''\left(\frac{x}{\lambda}\right)-x(k-1)\left[G'\left(\frac{x}{\lambda}\right)\right]^2+2\lambda\left[1-G\left(\frac{x}{\lambda}\right)\right]G'\left(\frac{x}{\lambda}\right)\right\}.$$

The derivatives of F are evaluated at $x=l_\beta(\lambda)$ with $\lambda=\hat{\lambda}_n$. Instead of the actual value of $\sigma^2(\hat{\lambda}_n)$ we use estimate of it.

4 Application to Half-Logistic Distribution

Consider a member of the scale family of distributions, namely half-logistic distribution with scale parameter λ. The cdf of X is

$$F(x;\lambda)=1-\left[\frac{2e^{-x/\lambda}}{1+e^{-x/\lambda}}\right]^k \qquad x\geq 0,\ \lambda>0.$$

The pdf of X is

$$f(x;\lambda)=\frac{k}{\lambda}\frac{2^k e^{-kx/\lambda}}{\left(1+e^{-x/\lambda}\right)^{k+1}} \qquad x\geq 0,\ \lambda>0.$$

4.1 Maximum Likelihood Estimation

The complete log-likelihood function for half-logistic distribution with scale parameter λ from equation (1) is

$$L_c=nlog(k)-nlog(\lambda)+\sum_{i=1}^{m}log\left[\frac{2e^{-x_i/\lambda}}{\left(1+e^{-x_i/\lambda}\right)^2}\right]+(k-1)\sum_{i=1}^{m}log\left[\frac{2e^{-x_i/\lambda}}{1+e^{-x_i/\lambda}}\right]$$
$$+\sum_{i=1}^{m}\sum_{j=1}^{R_i}log\left[\frac{2e^{-z_{ij}/\lambda}}{\left(1+e^{-z_{ij}/\lambda}\right)^2}\right]+(k-1)\sum_{i=1}^{m}\sum_{j=1}^{R_i}log\left[\frac{2e^{-z_{ij}/\lambda}}{1+e^{-z_{ij}/\lambda}}\right]. \qquad (10)$$

In order to obtain MLE of λ, we use EM algorithm due to Dempster et al. (1977). For the E step in EM algorithm we take Expectation of Z_{ij}. The derivative of L_c with respect to λ is taken for the M step, where

$$\frac{dL_c}{d\lambda}=-\frac{n}{\lambda}+\frac{k}{\lambda^2}\sum_{i=1}^{m}x_i-\frac{(k+1)}{\lambda^2}\sum_{i=1}^{m}\frac{x_ie^{-x_i/\lambda}}{1+e^{-x_i/\lambda}}+\frac{k}{\lambda^2}\sum_{i=1}^{m}R_ia(x_i,k,\lambda^0)$$
$$-\frac{(k+1)}{\lambda^2}\sum_{i=1}^{m}R_ib(x_i,k,\lambda^0). \qquad (11)$$

$$\text{where } \ a(x_i, k, \lambda) = E\left(Z_{ij}\right) \text{and } \ b(x_i, k, \lambda) = E\left[\frac{Z_{ij}e^{-Z_{ij}/\lambda}}{1+e^{-Z_{ij}/\lambda}}\right].$$

To solve this equation, we use Newton-Raphson method.
Reliability function at time t is

$$R(t) = \left[\frac{2e^{-t/\lambda}}{1+e^{-t/\lambda}}\right]^k \qquad t \geq 0, \ \lambda > 0.$$

The Maximum likelihood estimate of $R(t)$ is

$$\hat{R}_n(t) = \left[\frac{2e^{-t/\hat{\lambda}_n}}{1+e^{-t/\hat{\lambda}_n}}\right]^k \qquad t \geq 0.$$

4.2 Fisher Information

The observed information = complete information - missing information.

$$I_x(\lambda) = I_w(\lambda) - I_{w|x}(\lambda),$$

Consider log-likelihood function for n observations is

$$L = nlog(k) - nlog(\lambda) + \sum_{i=1}^{n} log\left[\frac{2e^{-x_i/\lambda}}{\left(1+e^{-x_i/\lambda}\right)^2}\right] + (k-1)\sum_{i=1}^{n} log\left[\frac{2e^{-x_i/\lambda}}{1+e^{-x_i/\lambda}}\right]. \quad (12)$$

Then complete information is

$$I_w(\lambda) = -E\left[\frac{d^2L}{d\lambda^2}\right] = -\frac{n}{\lambda^2} + \frac{2k}{\lambda^3}\sum_{i=1}^{n} E\left[X_i\right] + \frac{(k+1)}{\lambda^4}\sum_{i=1}^{n} E\left[\frac{X_i^2 e^{-X_i/\lambda}}{(1+e^{-X_i/\lambda})^2}\right]$$
$$-\frac{2(k+1)}{\lambda^3}\sum_{i=1}^{n} E\left[\frac{X_i e^{-X_i/\lambda}}{1+e^{-X_i/\lambda}}\right]. \quad (13)$$

and missing information is given by

$$I_{w|x}(\lambda) = \sum_{i=1}^{m} R_i I_{w|x}^{(i)}(\lambda) = -\sum_{i=1}^{m}\sum_{j=1}^{R_i} E_{Z|X}\left[\frac{d^2 log\left(f(Z_{ij}|x_i,\lambda)\right)}{d\lambda^2}\right]$$
$$= -\frac{n-m}{\lambda^2} + \frac{2k}{\lambda^3}\sum_{i=1}^{m}\sum_{j=1}^{R_i} E\left[Z_{ij}\right] + \frac{(k+1)}{\lambda^4}\sum_{i=1}^{m}\sum_{j=1}^{R_i} E\left[\frac{Z_{ij}^2 e^{-Z_{ij}/\lambda}}{(1+e^{-Z_{ij}/\lambda})^2}\right]$$
$$-\frac{2(k+1)}{\lambda^3}\sum_{i=1}^{m}\sum_{j=1}^{R_i} E\left[\frac{Z_{ij}e^{-Z_{ij}/\lambda}}{1+e^{-Z_{ij}/\lambda}}\right] - \frac{k}{\lambda^4}\sum_{i=1}^{m}\left[\frac{R_i x_i^2 e^{-x_i/\lambda}}{(1+e^{-x_i/\lambda})^2}\right]$$

$$+\frac{2k}{\lambda^3}\sum_{i=1}^{m}\left[\frac{R_i x_i e^{-x_i/\lambda}}{1+e^{-x_i/\lambda}}\right]-\frac{2k}{\lambda^3}\sum_{i=1}^{m}R_i x_i. \tag{14}$$

4.3 Confidence Interval and Tolerance Interval

Using equations (6) - (8) with $\hat{\sigma}^2(\hat{\lambda}_n)=\frac{1}{I_x(\hat{\lambda}_n)}$ and

$$\sigma^2(\hat{R}_n(t))\approx\left[\frac{kt}{\hat{\lambda}_n^2}\frac{\left(2e^{-t/\hat{\lambda}_n}\right)^k}{\left(1-e^{-t/\hat{\lambda}_n}\right)^{k+1}}\right]^2\sigma^2(\hat{\lambda}_n)$$

we construct confidence intervals for scale parameter and reliability function.

Let $l_\beta(\lambda)$ be the lower quantile of order β of the cdf $F(x;\lambda)$. Then, we have

$$l_\beta(\lambda)=\lambda log\left[\frac{2-(1-\beta)^{1/k}}{(1-\beta)^{1/k}}\right],$$

Thus, an upper β-expectation Tolerance Interval for $F(x;\lambda)$ is obtained by

$$I_\beta=(0,l_\beta(\lambda)).$$

The maximum likelihood estimator of $l_\beta(\lambda)$ is given by

$$l_\beta(\hat{\lambda}_n)=\hat{\lambda}_n log\left[\frac{2-(1-\beta)^{1/k}}{(1-\beta)^{1/k}}\right],$$

yielding an approximate β- expectation tolerance interval as

$$\hat{I}_\beta=\left(0,\ l_\beta(\hat{\lambda}_n)\right).$$

The expectation of $\hat{I}_\beta$ can be obtained approximately using the approach suggested and given as,

$$E\left[F(I_\beta(\hat{\lambda}_n);\lambda)\right]\approx\beta-0.5\ F_{02}\ \sigma^2(\hat{\lambda}_n)+\frac{F_{01}\ \sigma^2(\hat{\lambda}_n)\ F_{11}}{F_{10}}, \tag{15}$$

where $F_{10}=\frac{k2^k}{\lambda}\frac{\left(e^{-x/\lambda}\right)^k}{\left(1+e^{-x/\lambda}\right)^{k+1}},\qquad F_{01}=-\frac{kx2^k}{\lambda^2}\frac{\left(e^{-x/\lambda}\right)^k}{\left(1+e^{-x/\lambda}\right)^{k+1}},$

$$F_{11}=\frac{k2^k}{\lambda^3}\frac{\left(e^{-x/\lambda}\right)^k}{\left(1+e^{-x/\lambda}\right)^{k+2}}\left[(kx-\lambda)-e^{-x/\lambda}(x+\lambda)\right],$$

and $F_{02}=-\frac{kx2^k}{\lambda^4}\frac{\left(e^{-x/\lambda}\right)^k}{\left(1+e^{-x/\lambda}\right)^{k+2}}\left[(kx-2\lambda)-e^{-x/\lambda}(x+2\lambda)\right].$

5 Simulation Study

A simulation study is carried out to investigate the performance of MLE, reliability estimate and confidence interval of the scale parameter of half-logistic distribution. We obtain estimate of bias and MSE for various progressively Type-II censoring scheme. Asymptotic confidence intervals based on the MLE and log-transformed MLE are compared through their confidence levels. The coverage of the β- expectation tolerance intervals is studied using simulation. Balakrishnan and Sandhu (1995) presented algorithm for sample generation from progressively Type-II censored scheme. This algorithm is used to generate progressively censored samples from half-logistic distribution of a k-unit series system.
Algorithm

1. Generate independently and identically distributed observations $(W_1, W_2, \ldots.., W_m)$ from U(0, 1).
2. For $(R_1, R_2, \ldots.., R_m)$ progressive Type-II censoring scheme, set $E_i = 1/(i + R_m + R_{m-1} + \ldots.. + R_{m-i+1})$ for $i = 1, 2, \ldots.., m$.
3. Set $V_i = W_i^{E_i}$ for $i = 1, 2, \ldots.., m$.
4. Set $U_i = 1 - V_m V_{m-1} \ldots.. V_{m-i+1}$ for $i = 1, 2, \ldots., m$. Then $(U_1, U_2, \ldots.., U_m)$ is the U(0, 1) progressively Type-II censored sample.
5. For the given value of the parameter λ, set

$$x_i = \lambda \, log \left[\frac{2 - (1 - U_i)^{1/k}}{(1 - U_i)^{1/k}} \right] \qquad \text{for } i = 1, 2, \ldots.., m. \tag{16}$$

Then $(x_1, x_2, \ldots., x_m)$ is the required progressively Type-II censored sample from the distribution of a k-unit series system with half-logistic distribution as the component life distribution In Table 1 scheme (a, b) stands for $R_1 = a$ and $R_2 = b$. Similar meaning holds for schemes described through completely specified vector, while scheme $(10, 4*0)$ means $R_1 = 10$ and rest four $R_i s$ are zero. i.e. $R_2 = R_3 = R_4 = R_5 = 0$. A simulation was carried out for 2-unit, 3-unit and 5-unit series system (i.e. k=2, 3 and 5) with $\lambda = 1$. EM algorithm and Newton-Raphson method are used to compute MLE. For each particular progressive censoring scheme, 10,000 sets of observations were generated. The bias, MSE, confidence levels with their standard errors (SE) for the corresponding confidence intervals for λ are displayed in Table 1 - 3 for k=2, 3 and 5 respectively. The bias, MSE, confidence levels with their SE for the confidence intervals for reliability function are displayed in Table 4 - 6 for k=2, 3 and 5 respectively. The simulated mean coverage and the estimated expectation of the tolerance interval are given in Table 7 - 9. ($^+$MSE and SE are given in parenthesis.)

Table 1: Bias, MSE^+, Confidence levels and its SE^+ for MLE (k=2)

n	m	Scheme No.	Scheme	Bias and MSE	Level and SE-MLE 90%	95%	Level and SE-log(MLE) 90%	95%
5	2	[1]	(3,0)	-0.0708	0.7481	0.7802	0.8411	0.8883
				(0.3379)	(0.0377)	(0.0343)	(0.0267)	(0.0198)
		[2]	(0,3)	-0.0654	0.7525	0.7824	0.8445	0.8934
				(0.3529)	(0.0372)	(0.0341)	(0.0263)	(0.0190)
		[3]	(1,2)	-0.0694	0.7540	0.7884	0.8489	0.8907
				(0.3497)	(0.0371)	(0.0334)	(0.0257)	(0.0195)
		[4]	(2,1)	-0.0642	0.7520	0.7868	0.8427	0.8900
				(0.3559)	(0.0373)	(0.0335)	(0.0265)	(0.0196)
15	5	[5]	(10, 4*0)	-0.0248	0.8339	0.8656	0.8727	0.9263
				(0.1425)	(0.0092)	(0.0078)	(0.0074)	(0.0046)
		[6]	(4*0, 10)	-0.0196	0.8325	0.8693	0.8807	0.9313
				(0.1624)	(0.0093)	(0.0076)	(0.0070)	(0.0043)
		[7]	(2,2,2,2,2)	-0.0205	0.8315	0.8643	0.8777	0.9303
				(0.1546)	(0.0093)	(0.0078)	(0.0072)	(0.0043)
	10	[8]	(5,9*0)	-0.0121	0.8652	0.9041	0.8902	0.9401
				(0.0702)	(0.0078)	(0.0058)	(0.0065)	(0.0038)
		[9]	(9*0,5)	-0.0141	0.8680	0.9037	0.8941	0.9434
				(0.0723)	(0.0076)	(0.0058)	(0.0063)	(0.0036)
		[10]	(3,2, 8*0)	-0.0134	0.8694	0.9057	0.8894	0.9368
				(0.0713)	(0.0076)	(0.0057)	(0.0066)	(0.0039)
20	10	[11]	(10, 9*0)	-0.0117	0.8669	0.9045	0.8863	0.9391
				(0.0705)	(0.0058)	(0.0043)	(0.005)	(0.0029)
		[12]	(9*0,10)	-0.0086	0.8686	0.9069	0.8936	0.9423
				(0.0767)	(0.0057)	(0.0042)	(0.0048)	(0.0027)
25	10	[13]	(15,9*0)	-0.0167	0.8679	0.9070	0.8927	0.9398
				(0.0693)	(0.0046)	(0.0034)	(0.0038)	(0.0023)
		[14]	(9*0,15)	-0.0161	0.8613	0.8973	0.8829	0.9356
				(0.0805)	(0.0048)	(0.0037)	(0.0041)	(0.0024)
		[15]	(5,5,5,7*0)	-0.0106	0.8641	0.9033	0.8893	0.9401
				(0.0733)	(0.0047)	(0.0035)	(0.0039)	(0.0023)
	15	[16]	(10, 14*0)	-0.0099	0.8792	0.9198	0.8952	0.9455
				(0.0464)	(0.0042)	(0.003)	(0.0038)	(0.0021)
		[17]	(14*0,10)	-0.0123	0.8745	0.9160	0.8935	0.9458
				(0.0499)	(0.0044)	(0.0031)	(0.0038)	(0.0021)
30	10	[18]	(20, 9*0)	-0.0079	0.8676	0.9070	0.8889	0.9366
				(0.0725)	(0.00380	(0.0028)	(0.0033)	(0.002)
		[19]	(9*0,20)	-0.0100	0.8637	0.8994	0.8888	0.9389
				(0.0844)	(0.0039)	(0.003)	(0.0033)	(0.0019)
	15	[20]	(15, 14*0)	-0.0089	0.8745	0.9142	0.8865	0.9400
				(0.0481)	(0.0037)	(0.0026)	(0.0034)	(0.0019)
		[21]	(14*0,15)	-0.0087	0.8792	0.9171	0.8940	0.9460
				(0.0523)	(0.0035)	(0.0025)	(0.0032)	(0.0017)
		[22]	(5,5,5,12*0)	-0.0073	0.8777	0.9219	0.8960	0.9437
				(0.0474)	(0.0036)	(0.0024)	(0.0031)	(0.0018)
	20	[23]	(10, 19*0)	-0.0040	0.8859	0.9281	0.8942	0.9452
				(0.0355)	(0.0034)	(0.0022)	(0.0032)	(0.0017)
		[24]	(19*0,10)	-0.0064	0.8891	0.9287	0.8973	0.9460
				(0.0366)	(0.0033)	(0.0022)	(0.0031)	(0.0017)
		[25]	(0,5,5,17*0)	-0.0064	0.8839	0.9273	0.8946	0.9449
				(0.0356)	(0.0034)	(0.0022)	(0.0031)	(0.0017)
50	20	[26]	(30,19*0)	-0.0058	0.8827	0.9245	0.8945	0.9440
				(0.0360)	(0.0021)	(0.0014)	(0.0019)	(0.0011)
		[27]	(19*0,30)	-0.0095	0.8773	0.9218	0.8892	0.9423
				(0.0411)	(0.0022)	(0.0014)	(0.002)	(0.0011)
	35	[28]	(15,34*0)	-0.0021	0.8920	0.9350	0.8950	0.9467
				(0.0207)	(0.0019)	(0.0012)	(0.0019)	(0.0010)
		[29]	(34*0,15)	-0.0054	0.8920	0.9346	0.8980	0.9473
				(0.0211)	(0.0019)	(0.0012)	(0.0018)	(0.0010)
		[30]	(5,5,5,32*0)	-0.0044	0.8898	0.9342	0.8962	0.9444
				(0.0205)	(0.0020)	(0.0012)	(0.0019)	(0.0011)

Table 2: Bias, MSE^{+}, Confidence levels and its SE^{+} for MLE (k=3)

n	m	Scheme No.	Scheme	Bias and MSE	Level and SE-MLE 90%	95%	Level and SE-log(MLE) 90%	95%
5	2	[1]	(3,0)	-0.0492	0.7498	0.7796	0.8368	0.8927
				(0.3704)	(0.0375)	(0.0344)	0.0273	(0.0192)
		[2]	(0,3)	-0.0535	0.7506	0.7858	0.8496	0.8980
				(0.3822)	(0.0374)	(0.0337)	(0.0256)	(0.0183)
		[3]	(1,2)	-0.0356	0.7606	0.7934	0.8535	0.9016
				(0.3921)	(0.0364	(0.0328)	(0.0250)	(0.0177)
		[4]	(2,1)	-0.0535	0.7549	0.7849	0.8443	0.8912
				(0.3774)	(0.0370)	(0.0338	(0.0263)	(0.0194)
15	5	[5]	(10, 4*0)	-0.0265	0.8251	0.8630	0.8742	0.9228
				(0.1503)	(0.0096)	(0.0079)	(0.0073)	(0.0047)
		[6]	(4*0, 10)	-0.0210	0.8300	0.8662	0.8787	0.9272
				(0.1705)	(0.0094)	(0.0077)	(0.0071)	(0.0045)
		[7]	(2,2,2,2,2)	-0.0271	0.8284	0.8612	0.8767	0.9246
				(0.1635)	(0.0095)	(0.0080)	(0.0072)	(0.0046)
	10	[8]	(5,9*0)	-0.0107	0.8658	0.9070	0.8922	0.9408
				(0.0733)	(0.0077)	(0.0056)	(0.0064)	(0.0037)
		[9]	(9*0,5)	-0.0103	0.8657	0.9024	0.8868	0.9396
				(0.0794)	(0.0078	(0.0059)	(0.0067)	(0.0038)
		[10]	(3,2, 8*0)	-0.0117	0.8685	0.9042	0.8905	0.9390
				(0.0719)	(0.0076)	(0.0058)	(0.0065)	(0.0038)
20	10	[11]	(10, 9*0)	-0.0136	0.8676	0.9055	0.8905	0.9426
				(0.0720)	(0.0057)	(0.0043)	(0.0049)	(0.0027)
		[12]	(9*0,10)	-0.0120	0.8653	0.9043	0.8924	0.9421
				(0.0818)	(0.0058)	(0.0043)	(0.0048)	(0.0027)
25	10	[13]	(15,9*0)	-0.0151	0.8612	0.8983	0.8815	0.9325
				(0.0756)	(0.0048)	(0.0037)	(0.0042)	(0.0025)
		[14]	(9*0,15)	-0.0098	0.8644	0.9023	0.8889	0.9385
				(0.0859)	(0.0047)	(0.0035)	(0.0040)	(0.0023)
		[15]	(5,5,5,7*0)	-0.0126	0.8639	0.9013	0.8875	0.9359
				(0.0764)	(0.0047)	(0.0036)	(0.0040)	(0.0024)
	15	[16]	(10, 14*0)	-0.0100	0.8714	0.9141	0.8881	0.9384
				(0.0493)	(0.0045)	(0.0031)	(0.0040)	(0.0023)
		[17]	(14*0,10)	-0.0098	0.8755	0.9121	0.8903	0.9407
				(0.0545)	(0.0044)	(0.0032)	(0.0039)	(0.0022)
30	10	[18]	(20, 9*0)	-0.0139	0.8649	0.9041	0.8878	0.9385
				(0.0737)	(0.0039)	(0.0029)	(0.0033)	(0.0019)
		[19]	(9*0, 20)	-0.0045	0.8666	0.9014	0.8877	0.9377
				(0.0894)	(0.0039)	(0.0030)	(0.0033)	(0.0019)
	15	[20]	(15, 14*0)	-0.0104	0.8766	0.9156	0.8893	0.9419
				(0.0493)	(0.0036)	(0.0026)	(0.0033)	(0.0018)
		[21]	(14*0,15)	-0.0091	0.8715	0.9137	0.8876	0.9379
				(0.0563)	(0.0037)	(0.0026)	(0.0033)	(0.0019)
		[22]	(5,5,5,12*0)	-0.0110	0.8767	0.9158	0.8889	0.9419
				(0.0497)	(0.0036)	(0.0026)	(0.0033)	(0.0018)
	20	[23]	(10, 19*0)	-0.0084	0.8789	0.9245	0.8937	0.9424
				(0.0369)	(0.0035)	(0.0023)	(0.0032)	(0.0018)
		[24]	(19*0,10)	-0.0052	0.8813	0.9252	0.8942	0.9428
				(0.0395)	(0.0035)	(0.0023)	(0.0032)	(0.0018)
		[25]	(0,5,5,17*0)	-0.0043	0.8831	0.9257	0.8937	0.9437
				(0.0378)	(0.0034)	(0.0023)	(0.0032)	(0.0018)
50	20	[26]	(30,19*0)	-0.0052	0.8821	0.9243	0.8894	0.9426
				(0.0375)	(0.0021)	(0.0014)	(0.0020)	(0.0011)
		[27]	(19*0,30)	-0.0060	0.8839	0.9248	0.8955	0.9459
				(0.0438)	(0.0021)	(0.0014)	(0.0019)	(0.0010)
	35	[28]	(15,34*0)	-0.0043	0.8865	0.9317	0.8919	0.9441
				(0.0212)	(0.0020)	(0.0013)	(0.0019)	(0.0011)
		[29]	(34*0,15)	-0.0025	0.8944	0.9404	0.8998	0.9473
				(0.0223)	(0.0019)	(0.0011)	(0.0018)	(0.0010)
		[30]	(5,5,5,32*0)	-0.0028	0.8896	0.9364	0.8965	0.9449
				(0.0215)	(0.0020)	(0.0012)	(0.0019)	(0.0010)

Table 3: Bias, MSE$^+$, Confidence levels and its SE$^+$ for MLE (k=5)

n	m	Scheme No.	Scheme	Bias and MSE	Level and SE (MLE) 90%	95%	Level and SE (log(MLE)) 90%	95%
5	2	[1]	(3,0)	-0.05431 (0.3776)	0.7545 (0.0370)	0.7878 (0.0334)	0.8445 (0.0263)	0.8924 (0.0192)
		[2]	(0,3)	-0.0283 (0.4394)	0.7489 (0.0376)	0.7825 (0.0340)	0.8444 (0.0263)	0.8959 (0.0187)
		[3]	(1,2)	-0.0329 (0.4076)	0.7626 (0.0362)	0.7932 (0.0328)	0.8498 (0.0255)	0.9024 (0.0176)
		[4]	(2,1)	-0.0372 (0.4153)	0.7536 (0.0371)	0.7861 (0.0336)	0.8441 (0.0263)	0.8948 (0.0188)
15	5	[5]	(10, 4*0)	-0.0191 (0.1563)	0.8306 (0.0094)	0.8668 (0.0077)	0.8755 (0.0073)	0.9279 (0.0045)
		[6]	(4*0, 10)	-0.0097 (0.1875)	0.8273 (0.0095)	0.8608 (0.0080)	0.8750 (0.0073)	0.9271 (0.0045)
		[7]	(2,2,2,2,2)	-0.0211 (0.1758)	0.8252 (0.0096)	0.8570 (0.0082)	0.8703 (0.0075)	0.9209 (0.0049)
	10	[8]	(5,9*0)	-0.0138 (0.0761)	0.8668 (0.0077)	0.9050 (0.0057)	0.8928 (0.0064)	0.9384 (0.0039)
		[9]	(9*0,5)	-0.0107 (0.0842)	0.8629 (0.0079)	0.8959 (0.0062)	0.8868 (0.0067)	0.9375 (0.0039)
		[10]	(3,2, 8*0)	-0.0143 (0.0801)	0.8562 (0.0082)	0.8966 (0.0062)	0.8827 (0.0069)	0.9324 (0.0042)
20	10	[11]	(10, 9*0)	-0.0128 (0.0783)	0.8641 (0.0059)	0.9033 (0.0044)	0.8893 (0.0049)	0.9378 (0.0029)
		[12]	(9*0,10)	-0.0093 (0.0870)	0.8680 (0.0057)	0.9027 (0.0044)	0.8897 (0.0049)	0.9413 (0.0028)
25	10	[13]	(15,9*0)	-0.0134 (0.0777)	0.8651 (0.0047)	0.9032 (0.0035)	0.8893 (0.0039)	0.9365 (0.0024)
		[14]	(9*0,15)	-0.0133 (0.0870)	0.8682 (0.0046)	0.9025 (0.0035)	0.8927 (0.0038)	0.9419 (0.0022)
		[15]	(5,5,5,7*0)	-0.0079 (0.0797)	0.8670 (0.0046)	0.9058 (0.0034)	0.8930 (0.0038)	0.9400 (0.0023)
	15	[16]	(10, 14*0)	-0.0110 (0.0515)	0.8777 (0.0043)	0.9171 (0.0030)	0.8914 (0.0039)	0.9409 (0.0022)
		[17]	(14*0,10)	-0.0093 (0.0580)	0.8750 (0.0044)	0.9158 (0.0031)	0.8923 (0.0038)	0.9426 (0.0022)
30	10	[18]	(20, 9*0)	-0.0138 (0.0791)	0.8602 (0.0040)	0.8968 (0.0031)	0.8843 (0.0034)	0.9362 (0.0020)
		[19]	(9*0,20)	-0.0064 (0.0920)	0.8660 (0.0039)	0.9018 (0.0030)	0.8886 (0.0033)	0.9375 (0.0020)
	15	[20]	(15, 14*0)	-0.0097 (0.0517)	0.8782 (0.0036)	0.9188 (0.0025)	0.8932 (0.0032)	0.9419 (0.0018)
		[21]	(14*0,15)	-0.0022 (0.0578)	0.8819 (0.0035)	0.9234 (0.0024)	0.8991 (0.0030)	0.9468 (0.0017)
		[22]	(5,5,5,12*0)	-0.0095 (0.0517)	0.8808 (0.0035)	0.9204 (0.0024)	0.8950 (0.0031)	0.9427 (0.0018)
	20	[23]	(10, 19*0)	-0.0066 (0.0389)	0.8864 (0.0034)	0.9239 (0.0023)	0.8936 (0.0032)	0.9458 (0.0017)
		[24]	(19*0,10)	-0.0071 (0.0424)	0.8796 (0.0035)	0.9226 (0.0024)	0.8955 (0.0031)	0.9445 (0.0017)
		[25]	(0,5,5,17*0)	-0.0067 (0.0391)	0.8841 (0.0034)	0.9262 (0.0023)	0.8961 (0.0031)	0.9423 (0.0018)
50	20	[26]	(30,19*0)	-0.0117 (0.0389)	0.8801 (0.0021)	0.9221 (0.0014)	0.8947 (0.0019)	0.9436 0.0011
		[27]	(19*0,30)	-0.0057 (0.0447)	0.8840 (0.0021)	0.9238 (0.0014)	0.8939 (0.0019)	0.9455 (0.0010)
	35	[28]	(15,34*0)	-0.0059 (0.0228)	0.8806 (0.0021)	0.9316 (0.0013)	0.8891 (0.0020)	0.9434 (0.0011)
		[29]	(34*0,15)	-0.0030 (0.0242)	0.8936 (0.0019)	0.9378 (0.0012)	0.8979 (0.0018)	0.9468 (0.0010)
		[30]	(5,5,5,32*0)	-0.0022 (0.0219)	0.8887 (0.0020)	0.9416 (0.0011)	0.9035 (0.0017)	0.9511 (0.0009)

Table 4: Bias, MSE$^+$, Confidence levels and its SE$^+$ for R(t) (k=2)

n	m	Scheme No.	Scheme	Bias and MSE	Level and SE (MLE) 90%	95%
5	2	[1]	(3,0)	-0.1108 (0.0660)	0.7909 (0.0331)	0.8309 (0.0281)
		[2]	(0,3)	-0.1142 (0.0677)	0.7909 (0.0331)	0.8350 -0.0276
		[3]	(1,2)	-0.1088 (0.0657)	0.7963 (0.0324)	0.8391 (0.0270)
		[4]	(2,1)	-0.1182 (0.0687)	0.7861 (0.0336)	0.8267 (0.0287)
15	5	[5]	(10, 4*0)	-0.0472 (0.0246)	0.8634 (0.0079)	0.9116 (0.0054)
		[6]	(4*0, 10)	0.0541 (0.0282)	0.8580 (0.0081)	0.9067 (0.0056)
		[7]	(2,2,2,2,2)	-0.0499 (0.0270)	0.8599 (0.0080)	0.9077 (0.0056)
	10	[8]	(5,9*0)	-0.0229 (0.0108)	0.8826 (0.0069)	0.9372 (0.0039)
		[9]	(9*0,5)	-0.0260 (0.0117)	0.8846 (0.0068)	0.9308 (0.0043)
		[10]	(3,2, 8*0)	-0.0226 (0.0108)	0.8859 (0.0067)	0.9337 (0.0041)
20	10	[11]	(10, 9*0)	-0.0260 (0.0113)	0.8811 (0.0052)	0.9316 (0.0032)
		[12]	(9*0,10)	-0.0276 (0.0120)	0.8884 (0.0050)	0.9380 (0.0029)
25	10	[13]	(15,9*0)	-0.0239 (0.0109)	0.8864 (0.0040)	0.9359 (0.0024)
		[14]	(9*0,15)	-0.0279 (0.0130)	0.8795 (0.0042)	0.9312 (0.0026)
		[15]	(5,5,5,7*0)	-0.0237 (0.0110)	0.8847 (0.0041)	0.9345 (0.0024)
	15	[16]	(10, 14*0)	-0.0157 (0.0069)	0.8886 (0.0040)	0.9409 (0.0022)
		[17]	(14*0,10)	-0.0152 (0.0071)	0.8943 (0.0038)	0.9425 (0.0022)
30	10	[18]	(20, 9*0)	-0.0269 (0.0116)	0.8783 (0.0036)	0.9258 (0.0023)
		[19]	(9*0,20)	-0.0261 (0.0136)	0.8726 (0.0037)	0.9253 (0.0023)
	15	[20]	(15, 14*0)	-0.0158 (0.0069)	0.8897 (0.0033)	0.9394 (0.0019)
		[21]	(14*0,15)	-0.0191 (0.0081)	0.8841 (0.0034)	0.9373 (0.0020)
		[22]	(5,5,5,12*0)	-0.0155 (0.0069)	0.8921 (0.0032)	0.9422 (0.0018)
	20	[23]	(10, 19*0)	-0.0107 (0.0049)	0.8969 (0.0031)	0.9454 (0.0017)
		[24]	(19*0,10)	-0.0142 (0.0054)	0.8935 (0.0032)	0.9431 (0.0018)
		[25]	(0,5,5,17*0)	-0.0133 (0.0050)	0.8944 (0.0031)	0.9457 (0.0017)
50	20	[26]	(30,19*0)	-0.0119 (0.0051)	0.8903 (0.0020)	0.9410 (0.0011)
		[27]	(19*0,30)	-0.0159 (0.0060)	0.8906 (0.0019)	0.9390 (0.0011)
	35	[28]	(15,34*0)	-0.0069 (0.0028)	0.8969 (0.0018)	0.9446 (0.0010)
		[29]	(34*0,15)	-0.0076 (0.0029)	0.8934 (0.0019)	0.9479 (0.0010)
		[30]	(5,5,5,32*0)	-0.0069 (0.0028)	0.8924 (0.0019)	0.9440 (0.0011)

Table 5: Bias, MSE^+, Confidence levels and its SE^+ for R(t) (k=3)

n	m	Scheme No.	Scheme	Bias and MSE	Level and SE (MLE) 90%	95%
5	2	[1]	(3,0)	-0.0947 (0.0570)	0.7412 (0.0384)	0.7829 (0.0340)
		[2]	(0,3)	-0.0880 (0.0578)	0.7470 (0.0378)	0.7823 (0.0341)
		[3]	(1,2)	-0.0886 (0.0570)	0.7510 (0.0374)	0.7878 (0.0334)
		[4]	(2,1)	-0.0896 (0.0570)	0.7463 (0.0379)	0.7854 (0.0337)
15	5	[5]	(10, 4*0)	-0.0435 (0.0254)	0.8423 (0.0089)	0.8914 (0.0065)
		[6]	(4*0, 10)	-0.0455 (0.0287)	0.8302 (0.0094)	0.8784 (0.0071)
		[7]	(2,2,2,2,2)	-0.0456 (0.0275)	0.8334 (0.0093)	0.8828 (0.0069)
	10	[8]	(5,9*0)	-0.0247 (0.0128)	0.8723 (0.0074)	0.9193 (0.0049)
		[9]	(9*0,5)	-0.0247 (0.0136)	0.8706 (0.0075)	0.9166 (0.0051)
		[10]	(3,2, 8*0)	-0.0228 (0.0129)	0.8657 (0.0078)	0.9189 (0.0050)
20	10	[11]	(10, 9*0)	-0.0229 (0.0129)	0.8650 (0.0058)	0.9164 (0.0038)
		[12]	(9*0,10)	-0.0244 (0.0140)	0.8691 (0.0057)	0.9199 (0.0037)
25	10	[13]	(15,9*0)	-0.0234 (0.0130)	0.8638 (0.0047)	0.9140 (0.0031)
		[14]	(9*0,15)	-0.0244 (0.0146)	0.8675 (0.0046)	0.9146 (0.0031)
		[15]	(5,5,5,7*0)	-0.0228 (0.0131)	0.8703 (0.0045	0.9161 (0.0031)
	15	[16]	(10, 14*0)	-0.0149 (0.0085)	0.8772 (0.0043)	0.9282 (0.0027)
		[17]	(14*0,10)	-0.0174 (0.0093)	0.8764 (0.0043)	0.9290 nn(0.0026)
30	10	[18]	(20, 9*0)	-0.0240 (0.0127)	0.8693 (0.0038)	0.9219 (0.0024)
		[19]	(9*0,20)	-0.0249 (0.0151)	0.8606 (0.0040)	0.9133 (0.0026)
	15	[20]	(15, 14*0)	-0.015 (0.0085)	0.8795 (0.0035)	0.9286 (0.0022)
		[21]	(14*0,15)	-0.0157 (0.0092)	0.8836 (0.0034)	0.9325 (0.0021)
		[22]	(5,5,5,12*0)	-0.0166 (0.0087)	0.8776 (0.0036)	0.9279 (0.0022)
	20	[23]	(10, 19*0)	-0.0115 (0.0062)	0.8863 (0.0034)	0.9389 (0.0019)
		[24]	(19*0,10)	-0.0129 (0.0066)	0.8887 (0.0033)	0.9385 (0.0019)
		[25]	(0,5,5,17*0)	-0.0127 (0.0064)	0.8783 (0.0036)	0.9295 (0.0022)
50	20	[26]	(30,19*0)	-0.0106 (0.0064)	0.8860 (0.0020)	0.9359 (0.0012)
		[27]	(19*0,30)	-0.0127 (0.0074)	0.8809 (0.0021)	0.9323 (0.0013)
	35	[28]	(15,34*0)	-0.0078 (0.0036)	0.8872 (0.0020)	0.9387 (0.0012)
		[29]	(34*0,15)	-0.0072 (0.0038)	0.8872 (0.0020)	0.9408 (0.0011)
		[30]	(5,5,5,32*0)	-0.0065 (0.0035)	0.8909 (0.0019)	0.9392 (0.0011)

Table 6: Bias, MSE^{+}, Confidence levels and its SE^{+} for R(t) (k=5)

n	m	Scheme No.	Scheme	Bias and MSE	Level and SE (MLE) 90%	95%
5	2	[1]	(3,0)	-0.0348 (0.0340)	0.6993 (0.0421)	0.7390 (0.0390
		[2]	(0,3)	-0.0363 (0.0347)	0.7000 (0.0420)	0.7319 (0.0392)
		[3]	(1,2)	-0.0341 (0.0349)	0.7020 (0.0418)	0.7344 (0.0390)
		[4]	(2,1)	-0.0341 (0.0343)	0.7028 (0.0418)	0.7364 (0.0388)
15	5	[5]	(10, 4*0)	-0.0203 (0.0176)	0.8093 (0.0103)	0.8515 (0.0084)
		[6]	(4*0, 10)	-0.0165 (0.0197)	0.8024 (0.0106)	0.8437 (0.0088)
		[7]	(2,2,2,2,2)	-0.0193 (0.0198)	0.7933 (0.0109)	0.8396 (0.0090)
	10	[8]	(5,9*0)	-0.0126 (0.0100)	0.8493 (0.0085)	0.8907 (0.0065)
		[9]	(9*0,5)	-0.0133 (0.0109)	0.8360 (0.0091)	0.8808 (0.0070)
		[10]	(3,2, 8*0)	-0.0117 (0.0101)	0.8493 (0.0085)	0.8956 (0.0062)
20	10	[11]	(10, 9*0)	-0.0124 (0.0104)	0.8436 (0.0066)	0.8871 (0.0050)
		[12]	(9*0,10)	-0.0107 (0.0115)	0.8366 (0.0068)	0.8852 (0.0051)
25	10	[13]	(15,9*0)	-0.0133 (0.0102)	0.8450 (0.0052)	0.8931 (0.0038)
		[14]	(9*0,15)	-0.0116 (0.0112)	0.8465 (0.0052)	0.8919 (0.0039)
		[15]	(5,5,5,7*0)	-0.0123 (0.0105)	0.8414 (0.0053)	0.8900 (0.0039)
	15	[16]	(10, 14*0)	-0.0089 (0.0069)	0.8705 (0.0045)	0.9134 (0.0032)
		[17]	(14*0,10)	-0.0092 (0.0078)	0.8586 (0.0049)	0.9025 (0.0035)
30	10	[18]	(20, 9*0)	-0.0120 (0.0101)	0.8492 (0.0043)	0.8970 (0.0031)
		[19]	(9*0,20)	-0.0106 (0.0116)	0.8432 (0.0044)	0.8864 (0.0034)
	15	[20]	(15, 14*0)	-0.0095 (0.0070)	0.8617 (0.0040)	0.9108 (0.0027)
		[21]	(14*0,15)	-0.0087 0.0079	0.8570 (0.0041)	0.9054 (0.0029)
		[22]	(5,5,5,12*0)	-0.0069 (0.0072)	0.8596 (0.0040)	0.9079 (0.0028)
	20	[23]	(10, 19*0)	-0.0063 (0.0054)	0.8729 (0.0037)	0.9186 (0.0025)
		[24]	(19*0,10)	-0.0069 (0.0058)	0.8705 (0.0038)	0.9182 (0.0025)
		[25]	(0,5,5,17*0)	-0.0065 (0.0055)	0.8717 (0.0037)	0.9168 (0.0025)
50	20	[26]	(30,19*0)	-0.0077 (0.0054)	0.8712 (0.0022)	0.9178 (0.0015)
		[27]	(19*0,30)	-0.0069 (0.0063)	0.8663 (0.0023)	0.9142 (0.0016)
	35	[28]	(15,34*0)	-0.0045 (0.0031)	0.8849 (0.0020)	0.9355 (0.0012)
		[29]	(34*0,15)	-0.0046 (0.0033)	0.8855 (0.0020)	0.9315 (0.0013)
		[30]	(5,5,5,32*0)	-0.0035 (0.0034)	0.8828 (0.0020)	0.9311 (0.0013)

Table 7: Simulated mean and estimated expectation of the approximate β- expectation tolerance interval for k=2

n	m	Scheme No.	Scheme	Simulated Mean 90%	95%	99%	Estimated Expectation 90%	95%	99%
5	2	[1]	(3,0)	0.7630	0.8175	0.8874	0.7916	0.8792	0.9685
		[2]	(0,3)	0.7612	0.8151	0.8843	0.7835	0.8738	0.9669
		[3]	(1,2)	0.7619	0.8163	0.8860	0.7851	0.8749	0.9672
		[4]	(2,1)	0.7628	0.8172	0.8869	0.7879	0.8767	0.9677
15	5	[5]	(10, 4*0)	0.8430	0.8975	0.9564	0.8584	0.9228	0.9817
		[6]	(4*0, 10)	0.8392	0.8935	0.9530	0.8518	0.9185	0.9804
		[7]	(2,2,2,2,2)	0.8407	0.8949	0.9542	0.8540	0.9199	0.9809
	10	[8]	(5,9*0)	0.8717	0.9247	0.9758	0.8798	0.9368	0.9860
		[9]	(9*0,5)	0.8700	0.9232	0.9748	0.8786	0.936	0.9857
		[10]	(3,2, 8*0)	0.8710	0.9242	0.9755	0.8797	0.9367	0.9860
20	10	[11]	(10, 9*0)	0.8716	0.9246	0.9757	0.8797	0.9367	0.9860
		[12]	(9*0,10)	0.8704	0.9232	0.9746	0.8774	0.9352	0.9855
25	10	[13]	(15,9*0)	0.8706	0.9240	0.9755	0.8796	0.9367	0.9859
		[14]	(9*0,15)	0.8668	0.9203	0.9729	0.8765	0.9347	0.9853
		[15]	(5,5,5,7*0)	0.8711	0.9240	0.9752	0.8791	0.9364	0.9859
	15	[16]	(10, 14*0)	0.8806	0.9330	0.9810	0.8865	0.9412	0.9873
		[17]	(14*0,10)	0.8787	0.9313	0.9801	0.8854	0.9405	0.9871
30	10	[18]	(20, 9*0)	0.8721	0.9248	0.9756	0.8796	0.9366	0.9859
		[19]	(9*0,20)	0.8676	0.9206	0.9729	0.8759	0.9342	0.9852
	15	[20]	(15, 14*0)	0.8803	0.9326	0.9807	0.8865	0.9412	0.9873
		[21]	(14*0,15)	0.8789	0.9313	0.9799	0.8849	0.9401	0.9870
		[22]	(5,5,5,12*0)	0.8811	0.9333	0.9811	0.8863	0.9411	0.9873
	20	[23]	(10, 19*0)	0.8862	0.9378	0.9836	0.8899	0.9434	0.9880
		[24]	(19*0,10)	0.8851	0.9370	0.9832	0.8893	0.9430	0.9879
		[25]	(0,5,5,17*0)	0.8855	0.9373	0.9834	0.8898	0.9434	0.9880
50	20	[26]	(30,19*0)	0.8855	0.9373	0.9834	0.8899	0.9434	0.9880
		[27]	(19*0,30)	0.8826	0.9348	0.9821	0.8882	0.9423	0.9877
	35	[28]	(15,34*0)	0.8920	0.9430	0.9865	0.8943	0.9462	0.9889
		[29]	(34*0,15)	0.8909	0.9422	0.9862	0.8939	0.9460	0.9888
		[30]	(5,5,5,32*0)	0.8914	0.9426	0.9863	0.8942	0.9462	0.9889

Table 8: Simulated mean and estimated expectation of the approximate β- expectation tolerance interval for k=3

n	m	Scheme No.	Scheme	Simulated Mean 90%	95%	99%	Estimated Expectation 90%	95%	99%
5	2	[1]	(3,0)	0.7696	0.8229	0.8900	0.7898	0.8772	0.9674
		[2]	(0,3)	0.7648	0.8184	0.8865	0.7807	0.8712	0.9655
		[3]	(1,2)	0.7713	0.8244	0.8915	0.7822	0.8722	0.9658
		[4]	(2,1)	0.7671	0.8209	0.8890	0.7849	0.8740	0.9664
15	5	[5]	(10, 4*0)	0.8423	0.8965	0.9550	0.8577	0.9221	0.9813
		[6]	(4*0, 10)	0.8386	0.8926	0.9516	0.8508	0.9175	0.9799
		[7]	(2,2,2,2,2)	0.8381	0.8923	0.9516	0.8528	0.9188	0.9803
	10	[8]	(5,9*0)	0.8724	0.9250	0.9755	0.8794	0.9364	0.9858
		[9]	(9*0,5)	0.8705	0.9232	0.9743	0.8778	0.9353	0.9854
		[10]	(3,2, 8*0)	0.8724	0.9250	0.9754	0.8793	0.9363	0.9858
20	10	[11]	(10, 9*0)	0.8718	0.9247	0.9754	0.8793	0.9364	0.9858
		[12]	(9*0,10)	0.8692	0.9221	0.9735	0.8766	0.9346	0.9852
25	10	[13]	(15,9*0)	0.8701	0.9231	0.9744	0.8793	0.9363	0.9857
		[14]	(9*0,15)	0.8687	0.9214	0.9730	0.8759	0.9341	0.9850
		[15]	(5,5,5,7*0)	0.8707	0.9235	0.9745	0.8788	0.936	0.9856
	15	[16]	(10, 14*0)	0.8803	0.9325	0.9803	0.8863	0.9409	0.9872
		[17]	(14*0,10)	0.8787	0.9310	0.9794	0.8849	0.9400	0.9869
30	10	[18]	(20, 9*0)	0.8711	0.9240	0.9749	0.8792	0.9363	0.9857
		[19]	(9*0,20)	0.8694	0.9218	0.9730	0.8753	0.9337	0.9849
	15	[20]	(15, 14*0)	0.8802	0.9324	0.9803	0.8863	0.9409	0.9872
		[21]	(14*0,15)	0.8784	0.9307	0.9792	0.8844	0.9397	0.9868
		[22]	(5,5,5,12*0)	0.8800	0.9323	0.9803	0.8861	0.9408	0.9871
	20	[23]	(10, 19*0)	0.8851	0.9369	0.9830	0.8898	0.9432	0.9879
		[24]	(19*0,10)	0.8852	0.9368	0.9828	0.8889	0.9427	0.9877
		[25]	(0,5,5,17*0)	0.8859	0.9375	0.9832	0.8897	0.9432	0.9879
50	20	[26]	(30,19*0)	0.8857	0.9373	0.9831	0.8897	0.9432	0.9879
		[27]	(19*0,30)	0.8835	0.9353	0.9820	0.8879	0.9420	0.9875
	35	[28]	(15,34*0)	0.8916	0.9426	0.9862	0.8942	0.9461	0.9888
		[29]	(34*0,15)	0.8917	0.9427	0.9862	0.8937	0.9458	0.9887
		[30]	(5,5,5,32*0)	0.8919	0.9428	0.9863	0.8941	0.9461	0.9888

Table 9: Simulated mean and estimated expectation of the approximate β- expectation tolerance interval for k=5

n	m	Scheme No.	Scheme	Simulated Mean 90%	95%	99%	Estimated Expectation 90%	95%	99%
5	2	[1]	(3,0)	0.7710	0.8250	0.8919	0.7884	0.8759	0.9665
		[2]	(0,3)	0.7688	0.8220	0.8888	0.7799	0.8702	0.9647
		[3]	(1,2)	0.7735	0.8267	0.8928	0.7811	0.8710	0.9650
		[4]	(2,1)	0.7706	0.824	0.8907	0.7834	0.8726	0.9655
15	5	[5]	(10, 4*0)	0.8459	0.8997	0.9568	0.8569	0.9214	0.9809
		[6]	(4*0, 10)	0.8414	0.8950	0.9528	0.8508	0.9173	0.9797
		[7]	(2,2,2,2,2)	0.8399	0.8939	0.9521	0.8523	0.9183	0.9800
	10	[8]	(5,9*0)	0.8722	0.9250	0.9754	0.8789	0.9360	0.9856
		[9]	(9*0,5)	0.8704	0.9231	0.9738	0.8772	0.9348	0.9852
		[10]	(3,2, 8*0)	0.8706	0.9235	0.9744	0.8789	0.9360	0.9856
20	10	[11]	(10, 9*0)	0.8715	0.9242	0.9747	0.8789	0.9360	0.9856
		[12]	(9*0,10)	0.8704	0.9231	0.9738	0.8762	0.9342	0.9850
25	10	[13]	(15,9*0)	0.8715	0.9243	0.9748	0.8788	0.9359	0.9855
		[14]	(9*0,15)	0.8689	0.9219	0.9732	0.8757	0.9339	0.9849
		[15]	(5,5,5,7*0)	0.8728	0.9253	0.9753	0.8784	0.9357	0.9855
	15	[16]	(10, 14*0)	0.8806	0.9328	0.9804	0.8860	0.9407	0.9871
		[17]	(14*0,10)	0.8792	0.9314	0.9795	0.8845	0.9397	0.9867
30	10	[18]	(20, 9*0)	0.8710	0.9238	0.9745	0.8788	0.9359	0.9855
		[19]	(9*0,20)	0.8699	0.9225	0.9734	0.8753	0.9336	0.9848
	15	[20]	(15, 14*0)	0.8808	0.9329	0.9804	0.8859	0.9407	0.9870
		[21]	(14*0,15)	0.8813	0.9330	0.9802	0.8841	0.9395	0.9867
		[22]	(5,5,5,12*0)	0.8810	0.9331	0.9806	0.8858	0.9406	0.9870
	20	[23]	(10, 19*0)	0.8858	0.9374	0.9831	0.8895	0.9430	0.9878
		[24]	(19*0,10)	0.8846	0.9363	0.9824	0.8886	0.9424	0.9876
		[25]	(0,5,5,17*0)	0.8857	0.9373	0.9831	0.8894	0.9430	0.9878
50	20	[26]	(30,19*0)	0.8843	0.9363	0.9826	0.8895	0.9430	0.9878
		[27]	(19*0,30)	0.8843	0.9360	0.9822	0.8878	0.9419	0.9874
	35	[28]	(15,34*0)	0.8911	0.9423	0.9860	0.8940	0.9460	0.9887
		[29]	(34*0,15)	0.8915	0.9425	0.9860	0.8935	0.9457	0.9886
		[30]	(5,5,5,32*0)	0.8924	0.9432	0.9864	0.8940	0.9460	0.9887

6 Real Data Application

Consider following real data which represents failure times, for a specific type of electrical insulation that was subjected to a continuously increasing voltage stress given by Lawless (2011).

12.3, 21.8, 24.4, 28.6, 43.2, 46.9, 70.7, 75.3, 95.5, 98.1, 138.6, 151.9.

According to Balakrishnan and Chan (1992), half-logistic distribution satisfactory fit to this data. We consider this data as outcome for lifetime for two unit series system. We use this data with three censoring schemes as (2,0,0,0), (0,0,0,2) and (1,1,0,0). We obtain reliability estimate for time period t=1. MLE of reliability estimate and its MSE is given in Table 10. We construct confidence interval based on MLE. These 90% and 95% confidence intervals and their lengths are presented in same Table.

Table 10: Bias, MSE^{+}, Confidence intervals and its length for R(t)

n	m	Scheme	Bias and MSE	90% C. I. and its length	95% C. I. and its length
6	4	(2,0,0,0)	-0.0084	(0.9689, 0.9970)	(0.9665, 0.9970)
			(0.000002)	0.0281	0.0305
		(0,0,0,2)	-0.0011	(0.9811, 0.9969)	(0.9796, 0.9984)
			(0.000075)	0.0158	0.0188
		(1,1,0,0)	-0.0049	(0.9747, 0.9957)	(0.9737, 0.9977)
			(0.00024)	0.021	0.024

Method of MLE using EM algorithm and confidence interval based on MLE of reliability function gives best performance for real data. Bias is small in case of conventional censoring scheme whereas MSE is small in case of progressive censoring scheme. Length of confidence interval is small in case of conventional censoring scheme.

7 Conclusion and Discussion

Simulation study results indicate that, the bias, MSE of the MLE and reliability estimate decrease with increase in sample size n and increase in the effective sample size m. Same trend is observed in case of SE of confidence level of confidence intervals. The MSE is relatively smaller for progressive Type-II censoring scheme as compared with conventional Type-II censoring scheme. Confidence levels of confidence interval using log-transformed MLE are better than the confidence levels of confidence interval using MLE. SE for confidence levels of confidence intervals using log-transformed MLE is smaller than SE for confidence levels of confidence intervals using MLE. Confidence levels of confidence intervals of reliability function are better for large sample size.

β-expectation tolerance interval shows good results. As sample size n and effective sample size m increases the estimated expectation and simulated mean approaches to nominal coverage. Estimated expectation and simulated mean have better coverage for progressive Type-II censoring scheme than conventional Type-II censoring scheme, for small sample size. As number of units in system (k) increases the simulated mean decreases, but estimated expectation increases.

EM algorithm method works well for small sample size and for smaller effective sample size. Overall both conventional Type-II censoring scheme and progressive Type-II censoring scheme give better results. The MSE of progressive Type-II censoring method is smaller than the MSE of conventional censoring method, while bias, confidence interval and β-expectation tolerance interval perform equally good for both the methods. The results reported in this paper can also be applied when k is replaced by any known positive real number.

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