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Some Aspects of Poisson, Mixture of Poisson and Generalized Poisson Distributions of order *k*

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In this paper, we have introduced a class of GPDs of order k upon using a class of Quasi Binomial distribution of order k and using Abels generalization of the Binomial formula from Riordan (1968). A few particular cases, like a class of GPD, GPD of order k and classical Poisson distribution have been studied. A mixture of Poisson and Generalized Poisson distribution along with their various inferential properties are discussed. Finally, the mixture of Poisson and GP distributions are fitted to some real life data and compared with classical Poisson distribution and GPD and the fittings of the mixture to the observed frequencies are found to be very good as judged by the corresponding chi-square values.

keywords: Distributions of order k; Mixtures of distributions; Generalized Poisson distributions of order k; a class of Quasi Binomial Distribution of order k; Limiting distributions.

1 Introduction

When the populations are supposed to be Poissonian having unequal mean and variance, then we expect Generalized Poisson distributions. In this type of situation, the probability of occurrence of an event does not remain constant but changes with time and / or previous occurrences.

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Consul and Jain (1973a,b) are the early researchers who developed a class of discrete distributions of the Poissonian type, known as Generalized Poisson distribution (GPD) with two parameters having the probability function,

$$p(k;a,z)=\begin{cases}\frac{1}{k!}a(a+kz)^{k-1}e^{-(a+kz)}; & k=0,1,2,...\\ 0; & k>m, \text{ when } z<0\end{cases} \tag{1}$$

where $a>0$, max $(-1,-\frac{a}{m})<z\leq 1$ and m, the largest positive integer for which $(a+zm)>0$, when $z<0$.

Since then different aspects of these distributions have been studied by Bowman and Shenton (1985), Consul (1988), Consul and Shoukri (1985), Jani (1987). The detailed review works on the book authored by Consul (1989) have been done by Kemp (1992), Olkin (1992), Shimzu (1992), etc.

Nandi et al. (1999) stating with definition of the exponential sums $k(a;s,z)$ mathematically, obtained a class of discrete distributions having the probability function,

$$p(k;a,s,z)=\frac{(a+kz)^{k+s}e^{kz}}{k!K(a;s;z)},\quad k=0,1,2,...;(a+kz)>0 \tag{2}$$

where the exponential sum is given by,

$$k(a;s,z)=\sum_{k\geqslant 0}\frac{1}{k!}(a+kz)^{k+s}e^{-kz} \tag{3}$$

They referred it as a class of GPDs.

2 A Class of Quasi Binomial Distribution of order *k*:

By using Abel's generalization of the binomial formula of Riordan (1968) and extending it to order k , we have defined a class of Quasi Binomial distribution of order k with parameters n, p and ϕ, and integers s and t, whose probability function is given by Gupta et al. (2008) as,

$$P_k(x;s,t;\phi)=\sum_{m=0}^{k-1}\sum_{\sum_{j=1}^{k}jx_j=n-m-kx}\binom{x_1+x_2+..x_k+x}{x_1,x_2,..x_k,x}\left(p+\phi\sum_{i=1}^{k}x_i\right)^{\sum_{i=1}^{k}x_i+s}\cdot$$
$$\cdot\left(1-p-\phi\sum_{i=1}^{k}x_i\right)^{n-\sum_{i=1}^{k}x_i+t}\cdot\frac{1}{B_k(n;p,q;s,t;\phi)} \tag{4}$$

where $x=0(1)n$, $p+q+n\phi=1$ and $-\frac{p}{n}<\phi<\frac{1-p}{n}$.
and,

$$B_k(n;p,q;s,t;\phi) = \sum_{m=0}^{k-1} \sum_{\sum_{j=1}^{k} jx_j = n-m-kx} \sum_{x=0}^{n} \binom{x_1+x_2+..x_k+x}{x_1,x_2,..x_k,x} \left(p+\phi\sum_{i=1}^{k} x_i\right)^{\sum_{i=1}^{k} x_i+s} \cdot$$

$$\cdot \left(q-\phi\sum_{i=1}^{k} x_i\right)^{n-\sum_{i=1}^{k} x_i+t} \tag{5}$$

By setting $n \to \infty$ and $\phi \to 0$, such that $np = \lambda$ and $n\phi = \psi$(finite), in equation 4, then the limiting distribution of QBDs of order k is the probability function of a class of GPDs of order k for $\lambda > 0$, and $|\psi| < 1$ is given by,

$$g_k(x;\lambda,s;\psi) = \sum_{x_1,x_2,..x_k \ni x_1+2x_2+...+kx_k=x} \frac{e^{-(\lambda+\psi\sum_{i=1}^{k} x_i)} \left(\lambda+\psi\sum_{i=1}^{k} x_i\right)^{\sum_{i=1}^{k} x_i+s}}{\Pi_{i=1}^{k} x_i! K_k(\lambda,s;\psi)} \tag{6}$$

where,

$$K_k(\lambda,s;\psi) = \sum_{x_1,x_2,..x_k \geq 0} \frac{e^{-(\lambda+\psi\sum_{i=1}^{k} x_i)} \left(\lambda+\psi\sum_{i=1}^{k} x_i\right)^{\sum_{i=1}^{k} x_i+s}}{\Pi_{i=1}^{k} x_i!} \tag{7}$$

So, the definition of a class of GPDs of order k is as follows,

2.1 Definition:

A random variable 'X' is said to follow a class of GPDs of order k, when it assumes only non-negative values having parameters λ and ψ, such that $\lambda > 0$, and $|\psi| < 1$, having the probability function:

$$g_k(x;\lambda,s;\psi) = \sum_{x_1,x_2,..x_k \ni x_1+2x_2+...+kx_k=x} \frac{e^{-(\lambda+\psi\sum_{i=1}^{k} x_i)} \lambda \left(\lambda+\psi\sum_{i=1}^{k} x_i\right)^{\sum_{i=1}^{k} x_i+s}}{\Pi_{i=1}^{k} x_i! K_k(\lambda,s;\psi)} \tag{8}$$

where $K_k(\lambda,s;\psi)$ is defined in equation 7.
Substituting various values of s, one may obtain different GPDs of order k.

2.2 Particular Cases:

2.2.1

If s=-1, the probability function in equation 8 reduces to the probability function of GPD of order k as follows:

$$P(X=x)=\sum_{x_1,x_2,\ldots x_k \ni x_1+2x_2+\ldots+kx_k=x} \frac{e^{-(\lambda+\psi\sum_{i=1}^{k}x_i)}\lambda\left(\lambda+\psi\sum_{i=1}^{k}x_i\right)^{\sum_{i=1}^{k}x_i-1}}{\Pi_{i=1}^{k}x_i!} \quad (9)$$

where $(\lambda+\psi\sum_{i=1}^{k}x_i)>0$

2.2.2

Substituting k=1 in equation 9, it reduces to GPD I of Consul and Jain (1973a,b), whose probability function is as follows:

$$P(X=x)=\frac{e^{-(\lambda+x\psi)}\lambda(\lambda+x\psi)^{x-1}}{x!}; \quad x=0,1,2,...and \quad |\psi|<1 \quad (10)$$

2.2.3

If s=0, ψ=0 and k=1, then the probability function in equation 8 becomes common Poisson distribution of Johnson and Kotz (1969).

3 A Mixture distribution of Poisson and Generalized Poisson of order k:

Philippou et al. (1983a,b) studied the Poisson distribution of order k and accordingly they gave a definition of Poisson distribution of order k as follows:

A random variable **X** is said to have the Poisson distribution of order k with parameter λ, denoted by $P_k(\lambda)$, if its probability function is given by,

$$P(X=x)=\sum_{x_1,x_2,\ldots,x_k} e^{-k\lambda}\frac{\lambda^{x_1+x_2+\ldots+x_k}}{x_1!x_2!\ldots x_k!}; \quad x=0,1,2,... \quad (11)$$

where the summation is over all the non-negative integers x_1, x_2,...x_k such that $x_1+2x_2+...+kx_k=x$.

Gupta et al. (2008) studied the Generalized Poisson distribution of order k and their different inferential properties, whose probability function is given in equation 9.

Here we have found out the probability function of Mixture distribution of Poisson and Generalized Poisson of order k, given by,

$$P(X=x)=w\frac{e^{-k\lambda}\lambda^{x_1+x_2+x_3+...+x_k}}{x_1!x_2!x_3!...x_k!}+(1-w)\frac{e^{-(\lambda+\psi\sum_{i=1}^{k}x_i)}\lambda\left(\lambda+\psi\sum_{i=1}^{k}x_i\right)^{\sum_{i=1}^{k}x_i-1}}{\Pi_{i=1}^{k}x_i!} \quad (12)$$

where $(\lambda+\psi\sum_{i=1}^{k}x_i)>0$ and $0<w<1$

4 A Mixture of a Poisson and a Generalized Poisson Distribution

The probability function of mixture of a Poisson and a Generalized Poisson distribution is,

$$P_r(X=x)=w\frac{e^{-a}a^x}{x!}+(1-w)\frac{a(a+x\lambda)^{x-1}e^{-(a+x\lambda)}}{x!};\quad x=0,1,2,...$$

The pgf is

$$H(u)=we^{a(u-1)}+(1-w)e^{a(t-1)}, where \quad t=ue^{\lambda(t-1)}$$

and the r-th descending factorial moment is,

$$\mu_{(r)}{}'=wa^r+(1-w)e^{-r\lambda}\frac{K(a+r\lambda;r-1;\lambda)}{K(a;-1;\lambda)}$$

where

$$K(a;-1;\lambda)=e^a a^{-1}$$

and

$$K(a;0;\lambda)=\frac{e^a}{(1-\lambda)}$$

Putting r =1, we get

$$Mean=\mu_{(1)}{}'=\frac{a(1-w\lambda)}{(1-\lambda)},$$

Assuming $r=2$, we get

$$\mu_{(2)}{}'=\frac{1}{(1-\lambda)^3}[wa^2(1-\lambda)^3+a(1-w)(a-a\lambda+2\lambda-\lambda^2)]$$

and hence

$$Var(X)=\frac{1}{(1-\lambda)}[2aw-aw\lambda+2a^2w-2a^2w^2-a]$$

Estimators for the parameters w, a and λ of the mixture can be obtained by using the method of moments or by the method of maximum likelihood estimation. The recurrence relation for the probabilities is

$$P(x+1) = \frac{a}{x+1}\left[\frac{wa^x + (1-w)(a+\lambda+x\lambda)^x e^{-(\lambda+x\lambda)}}{wa^x + (1-w)a(a+x\lambda)^{x-1}e^{-x\lambda}}\right]P(x)$$

5 Graphical Representation of A Mixture Of a Poisson and a Generalized Poisson Distribution

To study the behavior of the mixture of a Poisson and a Generalized Poisson Distribution with varying values of a and λ and for fixed value of w, the probabilities for possible values of x are computed and a number of graphs for particular values of a, w and λ are shown in Figures 1 and 2.

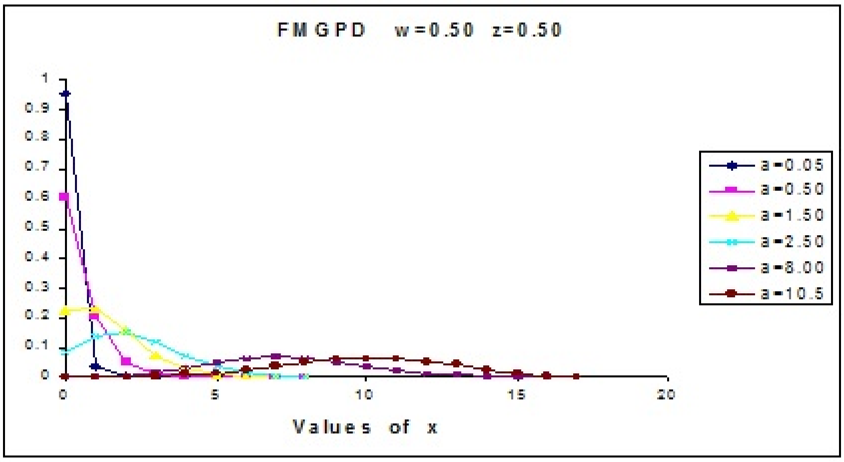

Figure 1: Graphs of probability distributions of mixture of a Poisson and a Generalized Poisson Distribution for w=0.50, $z = \lambda = 0.5$ and a=0.05, 0.50, 1.50, 2.50, 8.00, 10.5 respectively.

It is clear from the graphs of Figures 1 and 2 that for any given value of λ, the mixture of a Poisson and a Generalized Poisson Distribution is L shaped for values of $a < 1$, and as $'a'$ becomes larger the probability distribution acquires a larger span on the x-axis by moving to the right side, gradually losing its asymmetry and acquiring a bell-shaped form.

The graphs for a=8.0, λ=0.5, w=0.5; a=10.0, λ=0.05, w=0.5; a=10.0, λ=0.005, w=0.5; and a=10.5, λ=0.5, w=0.5 appear to be very close to the normal form.

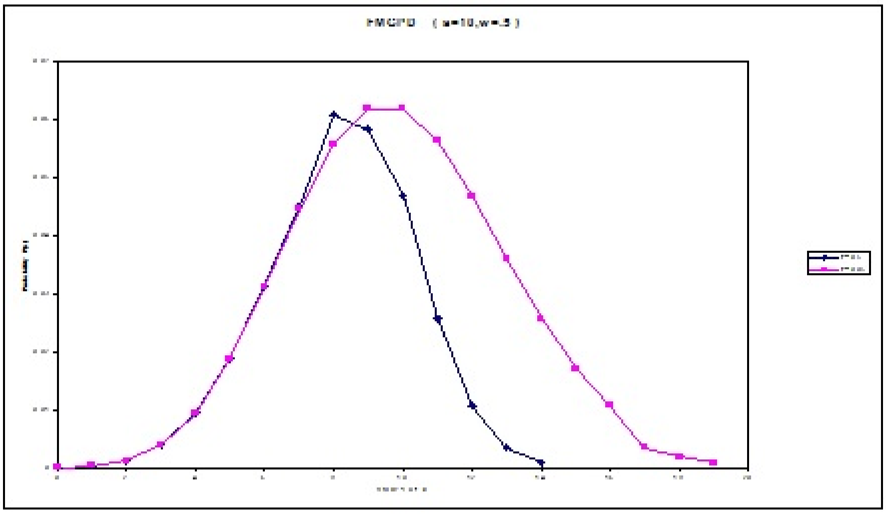

Figure 2: Graphs of probability distributions of mixture of a Poisson and a Generalized Poisson Distribution for a=10, w=0.5 and $z = \lambda$ =0.05, 0.005 respectively.

6 Maximum Likelihood Estimation of Mixture of a Poisson and a Generalized Poisson Distribution

The probability function of a mixture of Poisson and Generalized Poisson Distribution is given by,

$$P(X) = w\frac{e^{-a}a^x}{x!} + (1-w)\frac{a(a+x\lambda)^{x-1}e^{-(a+x\lambda)}}{x!}$$

Log likelihood function of this distribution is,

$$\log L = n\log w - na + \sum_{i=1}^{n} x_i \log a - \sum_{i=1}^{n} \log x_i! + n\log(1-w) + n\log a - \sum_{i=1}^{n}(a + x_i\lambda) + $$
$$+ \sum_{i=1}^{n}(x_i - 1)\log(a + x_i\lambda) - \sum_{i=1}^{n}\log x_i! \tag{13}$$

Differentiating equation 13 with respect to w, we get, $w = 0.5000$.

Now differentiating equation 13 with respect to a and λ, we can obtained two normal equations and solving these equations we get,

$$\hat{\lambda} = \frac{2(\bar{x} - a)}{\bar{x}}, \tag{14}$$

Table 1: The following data give the distribution of the catches of the Leech Herobdella in water samples

Leeches	Observed Frequency	Expected Frequency		
		Poisson	GPD	Poisson Λ GPD
0	58	45.56	55.81	58.01
1	25	37.16	26.66	25.85
2	13	15.14	11.53	11.69
3	2	4.11	4.98	2.68
4	2	0.83	2.18	2.46
5	1	0.13	0.97	1.31
6	1	0	0.43	0.62
7	0	0	0.27	0.21
8	1	0	0.12	0
9	0	0	0	0
Total	103	103	103	103
$\hat{a}$		0.8155	0.6127	0.5736
$\hat{\lambda}$			0.2486	0.5932
χ^2		8.3900	0.8014	0.8154
$d.f$		2	1	1

Mean = 0.8155

Putting the value of in one of the normal equations and applying the Newton - Raphson method we may obtained the value of a. Finally we can estimate the value of λ from equation 14.

7 Application

From the tables 1, 2 and 3, we studied the comparison of observed frequencies with that of expected frequencies.

Table 2: The following data from Lucy (1914) give the distribution of the number of days according to the number of deaths of women per day over 85 published in times during 1910-12.

Number of Deaths per Day	Observed Frequency	Expected Frequency		
		Poisson	GPD	Poisson Λ GPD
0	364	336.1	364.0	364.3
1	376	397.3	376.0	368.2
2	218	234.7	217.1	228.5
3	89	92.4	92.6	90.2
4	33	27.3	32.6	34.0
5	13	6.5	10.0	10.1
6	2	1.3	2.8	1.2
7	1	0.3	0.9	0.4
Total	1096	1096	1096	1096
$\hat{a}$		1.18157	1.10227	1.0765
$\hat{\lambda}$			0.16947	0.1778
χ^2		13.649	0.5346	2.1931
$d.f$		4	3	3

Mean = 1.18157

Table 3: The following data give the distribution of 400 squares of haemacytometer according to yeast cells (x) observed by 'Student'.

x	Observed Frequency	Expected Frequency		
		Poisson	GPD	Poisson Λ GPD
0	213	202.1	213.0	206.7
1	128	138.0	128.0	132.1
2	37	47.1	44.3	42.3
3	18	10.7	11.6	17.4
4	3	1.8	2.6	3.4
5	1	0.3	0.5	0.7
6	0	0	0	0
Total	400	400	400	400
$\hat{a}$		0.6825	0.63017	0.6602
$\hat{\lambda}$			0.0475	0.0653
χ^2		10.12	4.8281	1.1793
$d.f$		3	1	1

Mean = 0.6825

8 Conclusions

From the study of χ^2 values we conclude that in Tables 1 and 3, a mixture of Poisson distribution and Generalized Poisson Distribution gives the best fit to the observed frequencies.

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