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Heteroschedasticity in survey data and model selection based on weighted Schwarz-bayesian information criterion

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This paper proposed Weighted Schwarz Bayesian Information criterion for the purpose of selecting a best model from various competing models, when heteroschedasticity is present in the survey data. The authors found that the information loss between the true model and fitted models are equally weighted, instead of giving unequal weights. The computation of weights purely depends on the differential entropy of each sample observation and traditional Schwarz Bayesian information criterion was penalized by the weight function which comprised of the Inverse variance to mean ratio (VMR) of the fitted log-quantiles. The weighted Schwarz Bayesian information criterion was proposed in two versions based on the nature of the estimated error variances of the model namely Homogeneous and Heterogeneous WSBIC respectively. The proposed WSBIC outperforms the traditional information criterion of model selection and it leads to conduct a logical statistical treatment for selecting a best model. Finally this procedure was numerically illustrated by fitting 12 different types of stepwise regression models based on 44 independent variables in a BSQ (Bank service Quality) study.

keywords: Schwarz Bayesian information criterion, Weighted Schwarz Bayesian information criterion, Differential entropy, log-quantiles, Variance to mean ratio

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1 Introduction and Related work

Model selection is the task of selecting a statistical model from a set of candidate models, given data. Penalization is an approach to select a model that fits well with data which minimize the sum of empirical risk (FPE; Hirotugu Akaike, 1970; AIC; Hirotugu Akaike, 1973; Mallows' Cp; Colin L. Mallows, 1973). Many authors studied and proposed about penalties proportion to the dimension of model in regression, showing under various assumption sets that dimensionality-based penalties like Cp are asymptotically optimal (Ritei Shibata, 1981; Ker-Chau Li, 1987; Boris T. Polyak and A. B. Tsybakov, 1990) and satisfy non-asymptotic oracle inequalities (Yannick Baraud, 2000; Yannick Baraud, 2002; Barron, 1999; Lucien Birge and Pascal Massart, 2007). It is assumed that data can be heteroschedastic, but not necessary with certainty, (Arlot, 2010). Several estimators adapting to heteroschedasticity have been built thanks to model selection, (Xavier Gendre, 2008), but always assuming the model collection has a particular form. Past studies show that the general problem of model selection when the data are heteroschedastic can be solved only by cross-validation or resampling based procedures. This fact was recently confirmed, since resampling and V-fold penalties satisfy oracle inequalities for regressogram selection when data are heteroschedastic (Sylvain Arlot , 2009). Nevertheless, there is a significant increase of the computational complexity by adapting heteroschedasticity with resampling. The main goal of the paper is to propose a WSBIC (Weighted Schwarz Bayesian information criterion) if the problem of heteroschedasticity is present in the survey data. The derivation procedures of WSBIC and different versions of the criteria are discussed in the subsequent sections.

2 Homogeneous Weighted Schwarz-Bayesian Information Criterion

In the previous section the reviews of the model selection criteria used to select a model in the presence of heteroschedastic in the data are thoroughly discussed. This section deals with the presentation of the proposed Weighted Schwarz Bayesian information criterion. At first the authors highlighted the Schwarz Bayesian information criterion of a model based on log likelihood function and the blend of information theory is given as

$$SBIC = -2\log L(\widehat{\theta}/X) + k\log n \tag{1}$$

where $\widehat{\theta}$ is the estimated parameter, X is the data matrix,$L(\widehat{\theta}/X)$ is the maximized likelihood function and $k\log n$ is the penalty function which comprised of sample size *(n)* and no. of parameters *(k)* estimated in the fitted model. From (1), the shape of SBIC changes according to the nature of the penalty functions. Similarly, we derived a Weighted Schwarz Bayesian Information Criterion (WSBIC) based on the SBIC of a given model. Rewrite (1) as

$$SBIC = -2\log(\prod_{i=1}^{n} f(x_i/\widehat{\theta})) + k\log n$$

$$SBIC = -2\sum_{i=1}^{n} \log(f(x_i/\widehat{\theta})) + k \log n$$

$$SBIC = \sum_{i=1}^{n} (-2 \log f(x_i/\widehat{\theta}) + (k \log n/n)) \tag{2}$$

From (2), the quantity $-2 \log f(x_i/\widehat{\theta}) + (k \log n)/n)$ is the unweighted point wise information loss of an *ith* observation for a fitted model. The proposed WSBIC assured each point wise information loss should be weighted and it is defined as

$$WSBIC = \sum_{i=1}^{n} w_i(-2 \log f(x_i/\widehat{\theta}) + (k \log n/n)) \tag{3}$$

From (3), the weight of the point wise information loss shows the importance of the weightage that the model selector should give at the time of selecting a particular model. Here the problem is how the weights are determined?. The authors found, there is a link between the log quantiles of a fitted density function and the differential entropy. The following shows the procedure of deriving the weights.

Take mathematical expectation for (3), we get the expected WSBIC as

$$E(WSBIC) = \sum_{i=1}^{n} w_i(2E(-\log f(x_i/\widehat{\theta})) + (k \log n/n)) \tag{4}$$

where the term $E(-\log f(x_i/\widehat{\theta})) = \int_d -\log f(x_i/\widehat{\theta}) f(x_i/\widehat{\theta}) dx_i$ is the differential entropy of the *ith* observation and d is the domain of x_i, which is also referred as expected information in information theory. Now from (3) and (4), the variance of the WSBIC is given as

$$V(WSBIC) = 4E(\sum_{i=1}^{n} w_i(-\log f(x_i/\widehat{\theta}) - E(-\log f(x_i/\widehat{\theta}))))^2$$

$$\begin{aligned} V(WHQIC) = 4(&\sum_{i=1}^{n} w_i^2 V(-\log f(x_i/\widehat{\theta})) - 2\sum_{i=1}^{n}\sum_{j=1}^{n} E(((-\log f(x_i/\widehat{\theta}) \\ &-E(-\log f(x_i/\widehat{\theta}))((-\log f(x_j/\widehat{\theta}) - E(-\log f(x_j/\widehat{\theta})))) \end{aligned} \tag{5}$$

$$\begin{aligned} V(WHQIC) = 4(&\sum_{i=1}^{n} w_i^2 V(-\log f(x_i/\widehat{\theta})) - 2\sum_{i=1}^{n}\sum_{j=1}^{n} E(((-\log f(x_i/\widehat{\theta}) \\ &-E(-\log f(x_i/\widehat{\theta}))((-\log f(x_j/\widehat{\theta}) - E(-\log f(x_j/\widehat{\theta})))) \end{aligned} \tag{6}$$

From (5) $i \neq j$, the variance of the WSBIC was reduced by using iid property of the sample observation and it is given as

$$V(WSBIC) = 4(\sum_{i=1}^{n} w_i^2 V(-\log f(x_i/\widehat{\theta}))) \tag{7}$$

where $V(-\log f(x_i/\widehat{\theta})) = \int\limits_d E(-\log f(x_i/\widehat{\theta}) - E(-\log f(x_i/\widehat{\theta})))^2 f(x_i/\widehat{\theta}) dx_i$ is the variance of the fitted log quantiles which explains the variation between the actual and the expected point wise information loss. In order to determine the weights, the authors' wants to maximize $E(WSBIC)$ and minimize$V(WSBIC)$, because if the expected weighted information loss is maximum, then the variation between the actual weighted information and its expectation will be minimum. For this, maximize the difference (D) between the $E(WSBIC)$and $V(WSBIC)$which simultaneously optimize $E(WSBIC)$ and $V(WSBIC)$ then the D is given as

$$D = E(WSBIC) - V(WSBIC) \tag{8}$$

$$D = \sum_{i=1}^{n} w_i(2E(-\log f(x_i/\widehat{\theta})) + (k \log n/n)) - 4(\sum_{i=1}^{n} w_i^2 V(-\log f(x_i/\widehat{\theta})))$$

Using classical unconstrained optimization technique, maximize D with respect to the weights *(w)* by satisfying the necessary and sufficient conditions such as $\frac{\partial D}{\partial w_i} = 0$,$\frac{\partial^2 D}{\partial w_i^2} < 0$ and it is given as

$$\frac{\partial D}{\partial w_i} = 2E(-\log f(x_i/\widehat{\theta}) - w_i 8V(-\log f(x_i/\widehat{\theta})) = 0 \tag{9}$$

$$\frac{\partial^2 D}{\partial w_i^2} = -8V(-\log f(x_i/\widehat{\theta})) < 0 \tag{10}$$

By solving (9), we get the unconstrained weights as

$$w_i = \frac{E(-\log f(x_i/\widehat{\theta})}{4V(-\log f(x_i/\widehat{\theta}))} \tag{11}$$

From (9) and (10), it is impossible to use the second derivative Hessian test to find the absolute maximum or global maximum of the function D with respect to w_i,because the cross partial derivative $\partial^2 D/\partial w_i \partial w_j$ is 0 and w_i is not existing in $\partial^2 D/\partial w_i^2$. Hence the function D achieved the local maximum or relative maximum at the point w_i.Then from (11) rewrite the expectation and variance in terms of the integral representation as

$$w_i = \frac{\int\limits_d -\log f(x_i/\widehat{\theta}) f(x_i/\widehat{\theta}) dx_i}{4 \int\limits_d E(-\log f(x_i/\widehat{\theta}) - E(-\log f(x_i/\widehat{\theta})))^2 f(x_i/\widehat{\theta}) dx_i}$$

The equation (11), can also be represented in terms of VMR of fitted log quantiles and it is given as

$$w_i = \frac{1}{4VMR(-\log f(x_i/\widehat{\theta}))} \tag{12}$$

where $VMR(-\log f(x_i/\widehat{\theta})) = \frac{V(-\log f(x_i/\widehat{\theta}))}{E(-\log f(x_i/\widehat{\theta}))}$ is the variance to mean ratio. From (11) and (12), the maximum likelihood estimate $\widehat{\theta}$is same for all sample observations and the entropy, variance of the fitted log-quantiles are same for all i. Then $w_i = w$, then (3) becomes

$$WSBIC = w\sum_{i=1}^{n}(-2\log f(x_i/\widehat{\theta}) + (k\log n/n)) \tag{13}$$

where $w = \frac{E(-\log f(x/\widehat{\theta}))}{4V(-\log f(x/\widehat{\theta}))}$ for all i and substitute in (13), we get the homogeneous weighted version of the Weighted Schwarz Bayesian information criterion as

$$WSBIC = (\frac{E(-\log f(x/\widehat{\theta}))}{4V(-\log f(x/\widehat{\theta}))})\sum_{i=1}^{n}(-2\log f(x_i/\widehat{\theta}) + (k\log n/n))$$

$$WSBIC = \frac{\sum_{i=1}^{n}(-2\log f(x_i/\widehat{\theta}) + (k\log n/n))}{4VMR(-\log f(x/\widehat{\theta}))} \tag{14}$$

Combining (1) and (14) we get the final version of the homogeneous Weighted Schwarz Bayesian information criterion as

$$WSBIC = \frac{SBIC}{4VMR(-\log f(x/\widehat{\theta}))} \tag{15}$$

If a sample normal linear regression model is evaluated, with a single dependent variable (Y) with p regressors namely $X_{1i}, X_{2i}, X_{3i}, ...X_{pi}$ in matrix notation is given as

$$Y = X\beta + e \tag{16}$$

where $\underset{(nX1)}{Y}$ is the matrix of the dependent variable, $\underset{(kX1)}{\beta}$ is the matrix of beta coefficients or partial regression coefficients and $\underset{(nX1)}{e}$ is the residual followed normal distribution N (0,$\sigma_e^2 I_n$).From (15), the sample regression model should satisfy the assumptions of normality, homoschedasticity of the error variance and the serial independence property. Then the WSBIC of a fitted linear regression model is given as

$$WSBIC = \frac{SBIC}{4VMR(-\log f(Y/X, \widehat{\beta}, \widehat{\sigma_e^2}))} \tag{17}$$

where $\widehat{\beta}$,$\widehat{\sigma_e^2}$ are the maximum-likelihood estimates $SBIC = -2\log L(\widehat{\beta}, \widehat{\sigma_e^2}/Y, X) + k\log n$, $VMR(-\log f(Y/X, \widehat{\beta}, \widehat{\sigma_e^2}))$ is the variance to mean ratio of the fitted normal log quantiles and k is the no.of parameters estimated in the model (includes the Intercept and estimated error variance). From (17) VMR can be evaluated as

$$VMR(-\log f(Y/X,\widehat{\beta},\widehat{\sigma_e^2})) = \frac{V(-\log f(Y/X,\widehat{\beta},\widehat{\sigma_e^2}))}{E(-\log f(Y/X,\widehat{\beta},\widehat{\sigma_e^2}))} \tag{18}$$

$$VMR(-\log f(Y/X,\widehat{\beta},\widehat{\sigma_e^2})) = \frac{\int_{-\infty}^{+\infty} E(-\log f(Y/X,\widehat{\beta},\widehat{\sigma_e^2}) - E(-\log f(Y/X,\widehat{\beta},\widehat{\sigma_e^2})))^2 f(Y/X,\widehat{\beta},\widehat{\sigma_e^2})dY}{\int_{-\infty}^{+\infty} -\log f(Y/X,\widehat{\beta},\widehat{\sigma_e^2}) f(Y/X,\widehat{\beta},\widehat{\sigma_e^2})dY}$$

Where $f(Y/X,\widehat{\beta},\widehat{\sigma_e^2}) = \frac{1}{\sqrt{2\pi\widehat{\sigma_e^2}}} e^{-\frac{1}{2\widehat{\sigma_e^2}}(Y-\widehat{\beta}X)^2}, -\infty < Y < +\infty$ is the fitted normal density function and the expectation and variance of the quantity $-\log f(Y/X,\widehat{\beta},\widehat{\sigma_e^2})$ is given as

$$E(-\log f(Y/X,\widehat{\beta},\widehat{\sigma_e^2})) = \int_{-\infty}^{+\infty} (-\log f(Y/X,\widehat{\beta},\widehat{\sigma_e^2})) f(Y/X,\widehat{\beta},\widehat{\sigma_e^2})dy = \frac{1}{2}(1+\log(2\pi\widehat{\sigma_e^2})) \tag{19}$$

$$V(-\log f(Y/X,\widehat{\beta},\widehat{\sigma_e^2})) =$$

$$= \int_{-\infty}^{+\infty} E(-\log f(Y/X,\widehat{\beta},\widehat{\sigma_e^2}) - E(-\log f(Y/X,\widehat{\beta},\widehat{\sigma_e^2}))^2 f(Y/X,\widehat{\beta},\widehat{\sigma_e^2})dy = \frac{1}{2} \tag{20}$$

Substitute (19) and (20) in (18), then we get VMR for the fitted Normal log quantiles as

$$VMR(-\log f(Y/X,\widehat{\beta},\widehat{\sigma_e^2})) = \frac{1}{(1+\log(2\pi\widehat{\sigma_e^2}))} \tag{21}$$

Substitute (21) in (17), we get

$$WSBIC = \frac{(1+\log(2\pi\widehat{\sigma_e^2}))}{4} SBIC \tag{22}$$

where

$$w = \frac{1}{4}(1+\log(2\pi\widehat{\sigma_e^2})) \tag{23}$$

From (19), WSBIC is the product of the weight and the traditional Schwarz Bayesian information criterion. The WSBIC incorporates the dispersion in the fitted normal log quantiles and weighs the point wise information loss equally, but not with the unit weights. The mono weighted Schwarz Bayesian information criterion works based on the assumption of the homoschedastic error variance. If it is heteroschedastic, then we get the variable weights and the procedures are discussed in the next section.

3 Heterogeneous Weighted Schwarz Bayesian Information Criterion

The homogeneous weighted Schwarz Bayesian information criterion is impractical due to the assumption of homoschedasticity of the error variance. If this assumption is violated, then the weights vary for each point wise information loss, but the estimation of heteroschedastic error variance based on maximum likelihood estimation is difficult (Corderio, 2008; Fisher, 1957). For this, the authors utilize the link between the maximum likelihood theory and Least squares estimation to estimate the heteroschedastic error variance based on the following linear regression model.

Let the linear regression model with random error can be given as

$$\mathbf{Y} = \mathbf{X}\beta + \mathbf{e} \tag{24}$$

where $\underset{(nX1)}{Y}$ is the matrix of the dependent variable, $\underset{(kX1)}{\beta}$ is the matrix of beta coefficients or partial regression coefficients and $\underset{(nX1)}{e}$ is the residual followed normal distribution N $(0,\sigma_e^2 I_n)$.From (24), the fitted model with estimates as

$$\widehat{\mathbf{Y}} = \mathbf{X}\widehat{\beta} \tag{25}$$

$$\widehat{\beta} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y} \tag{26}$$

Substitute $\underset{(kX1)}{\widehat{\beta}}$ in (26), we get

$$\widehat{\mathbf{Y}} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}$$

$$\widehat{\mathbf{Y}} = \mathbf{H}\mathbf{Y} \tag{27}$$

From (27), the estimated $\underset{(nX1)}{\widehat{Y}}$ is predicted by $\underset{(nX1)}{Y}$ based on the matrix $\underset{(nXn)}{H} = X(X'X)^{-1}X'$, technically called as Hat matrix. Combine (23) and (27), we get a compact form of the random errors in terms of the hat matrix and it is given as

$$\mathbf{e} = \mathbf{Y} - \widehat{\mathbf{Y}} \tag{28}$$

$$\mathbf{e} = \mathbf{Y} - \mathbf{X}\widehat{\beta}$$

$$\mathbf{e} = \mathbf{Y} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}$$

$$\mathbf{e} = \mathbf{Y}(\mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}')$$

$$\mathbf{e} = (\mathbf{I} - \mathbf{H})\mathbf{Y} \tag{29}$$

From (29), the random errors are the product of actual value of $\underset{(nX1)}{Y}$ and the residual operator $(I-H)$. Myers and Montgomery (1997) proved the magical properties of the residual operator matrix as idempotent and symmetric. Based on the properties they derived the variance-co-variance matrix of the random errors as

$$\sum = \sigma_e^2(\mathbf{I}-\mathbf{H}) \tag{30}$$

where $\sum$ is the variance-covariance matrix of the errors and σ_e^2 is the homoschedastic error variance of a linear regression model. The authors utilize the least square estimate of the variance-covariance matrix of the error and found the link between the heteroschedastic and homoschedastic error variance. From (30), the estimate of $\sum$ is given as

$$\widehat{\sum} = \widehat{s_e^2}(\mathbf{I}-\mathbf{H}) \tag{31}$$

From (31), compare the diagonal elements of both sides, we get the estimated unbiased heteroschedastic error variance as

$$\widehat{s_{e_i}^2} = \widehat{s_e^2}(1-h_{ii}) \tag{32}$$

where $\widehat{s_{e_i}^2}, \widehat{s_e^2}$ are the unbiased estimates of heteroschedastic, homoschedastic error variance and h_{ii} is the leading diagonal elements of the hat matrix, sometimes called as centered leverage values. We know that the least squares estimates of error variance is unbiased and estimation of error variance based on maximum likelihood estimation theory is biased (Greene and William H. Greene, 2011), so the authors remove the unbiaseness in the least squares estimate of the error variance and convert it as biased estimated, which is equal to the maximum likelihood estimates. From (32), it can be rewrite as

$$(\frac{n-k}{n})\widehat{s_{e_i}^2} = (\frac{n-k}{n})\widehat{s_e^2}(1-h_{ii}) \tag{33}$$

$$\widehat{\sigma_{e_i}^2} = \widehat{\sigma_e^2}(1-h_{ii}) \tag{34}$$

From (34), the least squares estimate of error variance is transformed into maximum likelihood estimate and this relationship between the estimated heteroschedastic and homoschedastic estimated error variance was used to find the heterogeneous weights in the WSBIC. Combine (34) with (23), we get the weights for *ith* point wise information loss in WSBIC under the assumption of the estimated error variances are heteroschedastic and it as follows

$$w_i = \frac{1}{4}(1+\log(2\pi\widehat{\sigma_{e_i}^2})) \tag{35}$$

$$w_i = \frac{1}{4}(1+\log(2\pi\widehat{\sigma_e^2}(1-h_{ii}))) \tag{36}$$

$$w_i = \frac{1}{4}(1 + \log(2\pi\widehat{\sigma_e^2}) + \log(1 - h_{ii})) \quad (37)$$

$$w_i = w + \frac{1}{4}\log(1 - h_{ii}) \quad (38)$$

From (38), the authors found the relationship between the variable weights with homogeneous weights and h_{ii} is the centered leverage values which always lies between the $p/n \leq h_{ii} \leq 1$, where p is the no.of regressors.Hence,the authors proved from (37), if the estimated error variance is homoschedastic, we can derive the heteroschedastic error variance based on the hat values. Moreover,the variable weights gave importance to the point wise information loss unequally which the WSBIC can be derived by combining (3) and (37) in terms of the linear regression model as

$$WSBIC = \frac{1}{4}\sum_i((1 + \log(2\pi\widehat{\sigma_e^2}(1 - h_{ii})))(-2\log f(\mathbf{Y}/\mathbf{X}, \widehat{\beta}, \widehat{\sigma_e^2}) + (k\log n/n))) \quad (39)$$

4 Results and Discussion

In this section, we will investigate the discrimination between the traditional HQIC and the proposed WHQIC on the survey data collected from BSQ (Bank Service Quality) study. The data comprised of 45 different attributes about the Bank and the data was collected from 102 account holders. A well-structured questionnaire was prepared and distributed to 125 customers and the questions were anchored at five point Likert scale from 1 to 5. After the data collection is over, only 102 completed questionnaires were used for analysis. The following table shows the results extracted from the analysis by using SPSS version 20. At first, the authors used, stepwise multiple regression analysis by utilizing 44 independent variables and a dependent variable. The results of the stepwise regression analysis with model selection criteria are visualized in the following Table 1 with results of subsequent analysis.

Table 1: Stepwise Regression Summary, Traditional SBIC and Weighted SBIC

Model	Model	1	2	3	4	5	6	7	8	9	10	11	12
Regression summary	K	3	4	5	6	7	8	9	10	11	12	13	12
Regression summary	EHEV	0.23	0.19	0.177	0.167	0.164	0.157	0.147	0.141	0.133	0.126	0.123	0.127
Regression summary	R2	0.188	0.331	0.377	0.41	0.441	0.489	0.525	0.56	0.59	0.61	**0.63**	0.63
Regression summary	F-ratio	23.08*	24.48*	19.75*	16.84*	15.13*	15.14*	14.81*	15.08*	15.18*	14.54*	14.18*	15.46*
Regression summary	SBIC	166.52	155.96	157.97	161.66	168.94	173.41	177.62	183.68	188.15	192.50	200.85	194.24
Homogeneous WSBIC	MAX(D)	-26.33	-15.63	-10.63	-6.48	-3.62	-0.37	2.95	5.55	8	10.04	11.77	10.03
Homogeneous WSBIC	E(WSBIC)	-2.45	2.05	4.99	7.54	9.91	12.05	13.76	15.4	16.59	17.53	18.81	17.68
Homogeneous WSBIC	V(WSBIC)	23.88	17.68	15.62	14.01	13.54	12.41	10.81	9.85	8.59	7.5	7.04	7.65
Homogeneous WSBIC	W	0.34	0.29	0.27	0.26	0.25	0.24	0.23	0.22	0.20	0.191	0.186	0.194
Homogeneous WSBIC	WSBIC	57.11	45.85	43.60	42.51	43.58	42.66	41.03	40.41	38.76	**36.76**	37.35	37.68
Heterogeneous WSBIC	MAX(D)	33.05	27.76	27.19	27.06	28.16	27.93	26.94	26.53	25.33	23.89	23.92	24.36
Heterogeneous WSBIC	E(WSBIC)	56.66	44.82	41.94	40.09	40.45	38.79	36.19	34.65	32.13	29.48	**28.99**	30.14
Heterogeneous WSBIC	V(WSBIC)	23.61	17.06	14.75	13.03	12.28	10.86	9.25	8.12	6.8	5.59	**5.07**	5.77
Heterogeneous WSBIC	WSBIC	56.55	44.99	42.23	40.65	41.22	39.57	37.31	35.86	33.67	31.197	**30.71**	31.782

*p-value <0.01 SBIC- Schwarz Bayesian Information Criterion, EHEV-Estimated homoschedastic error variance, MAX (D)-Maximized difference, W-Weights, E (WSBIC)-Expectation of weighted Schwarz Bayesian information criterion, V (WSBIC)-Variance of weighted Schwarz Bayesian information criterion.

Table 2: Estimated Heteroschedastic error variance of Models

Observation	Model 1	Model 2	Model 3	Model 4	Model 5	Model 6	Model 7	Model 8	Model 9	Model 10	Model 11	Model 12
1	.22981	.18738	.17348	.16360	.16046	.15234	.13420	.12727	.12003	.11297	.11057	.11440
2	.22981	.18738	.17348	.16264	.15952	.15095	.14135	.13423	.12679	.11951	.11619	.11999
3	.22467	.18347	.16254	.15310	.14459	.13776	.12918	.12284	.11588	.10883	.10310	.11838
4	.22991	.18859	.17448	.16416	.16014	.15206	.14199	.13398	.12649	.11934	.11642	.12042
5	.22991	.17892	.16491	.15443	.15038	.14309	.13262	.12685	.11981	.10881	.10433	.10867
6	.22981	.18827	.17475	.16448	.15937	.15076	.14175	.13454	.12645	.11899	.11584	.11960
7	.22497	.18471	.17095	.16045	.15718	.14937	.13993	.13398	.12651	.11895	.11600	.11975
8	.22981	.18738	.17375	.16419	.15932	.15061	.14165	.13550	.12730	.11667	.10840	.11422
9	.22497	.18265	.16946	.15964	.15648	.14510	.13494	.12921	.12201	.10846	.10526	.10938
10	.22467	.18347	.16991	.15881	.15544	.14802	.13835	.13149	.12021	.11230	.10886	.11352
11	.22981	.18827	.16815	.15095	.14775	.13967	.13111	.12133	.11155	.10502	.10226	.10728
12	.22981	.18738	.17375	.16419	.15257	.14322	.13462	.12866	.12089	.11374	.09768	.10472
13	.22991	.18710	.17316	.16299	.15866	.15012	.14114	.13491	.12725	.11929	.11556	.12110
14	.22991	.18859	.17448	.16416	.16058	.15289	.14302	.13363	.12348	.11477	.11233	.11735
15	.22991	.18859	.17448	.16416	.16014	.15206	.14199	.13560	.12798	.11227	.10986	.11365
16	.22981	.18738	.16598	.15603	.15129	.14400	.13465	.12850	.12127	.11186	.10879	.11968
17	.22991	.18710	.17316	.16299	.15866	.15012	.14114	.13491	.10853	.09876	.09664	.09976
18	.22991	.18710	.17356	.16379	.15504	.14768	.13877	.13290	.12481	.11716	.11467	.11872
19	.21449	.17511	.16169	.15283	.14924	.14179	.13291	.12687	.11879	.11209	.10862	.11217
20	.22991	.18859	.17448	.16416	.15591	.14859	.13857	.13271	.12276	.11450	.11205	.11733
21	.22991	.18859	.17448	.16416	.16058	.15289	.14302	.13693	.12686	.11807	.11513	.11953
22	.22991	.18859	.16825	.15898	.15593	.14812	.13929	.12641	.11932	.11243	.10695	.11095
23	.22981	.18827	.17412	.16412	.15786	.14860	.13826	.13240	.12494	.11719	.11446	.11835
24	.22991	.18710	.16553	.15522	.14895	.14184	.13248	.12661	.11523	.10828	.10444	.11322
25	.22991	.17892	.16631	.15719	.15415	.14674	.13766	.13125	.12357	.11483	.10908	.11433
26	.22991	.18859	.17498	.16501	.16176	.15371	.14457	.13702	.12796	.12075	.11634	.12241
27	.22981	.18827	.17412	.16412	.16095	.15317	.14340	.13721	.12925	.12171	.11860	.12244
28	.21449	.17538	.16245	.15377	.15003	.12835	.12010	.11501	.10733	.10130	.09919	.10338
29	.22991	.18710	.17356	.16250	.15856	.15107	.14176	.10486	.09848	.09258	.08888	.09195
30	.22981	.18738	.17375	.16419	.16097	.15286	.14377	.13761	.12879	.12138	.11882	.12268
31	.22991	.18859	.17498	.16501	.15826	.14949	.13660	.13038	.12231	.11541	.11215	.11577
32	.22991	.18859	.17448	.16416	.16014	.15206	.14256	.13642	.12868	.12121	.11679	.12131
33	.22991	.18859	.17498	.16501	.16176	.15371	.14205	.13523	.12603	.11840	.11389	.12076
34	.22991	.18859	.17498	.15697	.15203	.13543	.12014	.11114	.10450	.09726	.09516	.09835
35	.22991	.18859	.16825	.15898	.15345	.14609	.13737	.12947	.11960	.11183	.09419	.09838
36	.22991	.18710	.17316	.16272	.15944	.15107	.14140	.13416	.12666	.11678	.11337	.11703
37	.22467	.18378	.17062	.16067	.15723	.14910	.14004	.13288	.12465	.11757	.11474	.11844

Table 3: Estimated Heteroschedastic error variance of Models

Observation	Model 1	Model 2	Model 3	Model 4	Model 5	Model 6	Model 7	Model 8	Model 9	Model 10	Model 11	Model 12
38	.22991	.18859	.17448	.16407	.15988	.15134	.14117	.13460	.12662	.11876	.11547	.11920
39	.21449	.17511	.16169	.15283	.14929	.14219	.13298	.12507	.11732	.11056	.10716	.11067
40	.22991	.18710	.17356	.16250	.15856	.14947	.14045	.13270	.12466	.11668	.11330	.11733
41	.22991	.18710	.17356	.16379	.15998	.15143	.14244	.13635	.12771	.11769	.11511	.11943
42	.22467	.18347	.16991	.16046	.15007	.14124	.13227	.12465	.11560	.10196	.09371	.09681
43	.22991	.18710	.17356	.16250	.15418	.14665	.13733	.13132	.12337	.11473	.11216	.11716
44	.22467	.18378	.16985	.16035	.15717	.14947	.13349	.12694	.11789	.11100	.10851	.11396
45	.22981	.18827	.17412	.16412	.15786	.14860	.13955	.13099	.12052	.11320	.10767	.11717
46	.22981	.18827	.17412	.16334	.16021	.15213	.14287	.13564	.12747	.11915	.11544	.12034
47	.22991	.18859	.17498	.16407	.14813	.14106	.12937	.12324	.11584	.10448	.10168	.10533
48	.22981	.18738	.17375	.16221	.15907	.15139	.14189	.13552	.12664	.11926	.11642	.12018
49	.22991	.17892	.16491	.15554	.15234	.14401	.13519	.12817	.12102	.11418	.11118	.11707
50	.22991	.18859	.17448	.16407	.16060	.15264	.14190	.13547	.12750	.11989	.11696	.12076
51	.22991	.18710	.17356	.16250	.15856	.15107	.14084	.13413	.12558	.11574	.11282	.11704
52	.22991	.18859	.17498	.16407	.15700	.14963	.13881	.13282	.12448	.11548	.11158	.11588
53	.22991	.18710	.17356	.15224	.14452	.13708	.12866	.12180	.10843	.10231	.10018	.10680
54	.22991	.18859	.17448	.16416	.16058	.15289	.14314	.13684	.12916	.12191	.11933	.12354
55	.22497	.18265	.16894	.15864	.15462	.14703	.13800	.13211	.12311	.11572	.11291	.11877
56	.22991	.18710	.17316	.16299	.15866	.15012	.13913	.13261	.11911	.11187	.10889	.11257
57	.22991	.18859	.17448	.16407	.16060	.15264	.14190	.13397	.12618	.11859	.11513	.12018
58	.22991	.18859	.17448	.16416	.15591	.14647	.13608	.13033	.12308	.11608	.11366	.11781
59	.22981	.18738	.17375	.15126	.14547	.13782	.12840	.12188	.11503	.10628	.10404	.10810
60	.22497	.18471	.16446	.15290	.14962	.14133	.13293	.12676	.11176	.10383	.10148	.10578
61	.22991	.18859	.16825	.15702	.15400	.14658	.13568	.12818	.11869	.11169	.10934	.11951
62	.22981	.18827	.17412	.16334	.15053	.14128	.13222	.12556	.11760	.11054	.10800	.11219
63	.22981	.18738	.17375	.16221	.15713	.14786	.13711	.12257	.11440	.10774	.10477	.10831
64	.22981	.18738	.17375	.16419	.16097	.15286	.14121	.13309	.12486	.11776	.11531	.11949
65	.22467	.18347	.17005	.16085	.15586	.14832	.12940	.12261	.11480	.10561	.10337	.10766
66	.22497	.18265	.16946	.15964	.15648	.14510	.13494	.12921	.12201	.10846	.10526	.10938
67	.22497	.18265	.16894	.15864	.15462	.14703	.13005	.12308	.11520	.10510	.10236	.10628
68	.22981	.18738	.17375	.16221	.15907	.15058	.14139	.13488	.12547	.11753	.11502	.11874
69	.22467	.18378	.17062	.16067	.15723	.14949	.14017	.13315	.12537	.11624	.11357	.11731
70	.22467	.18378	.17062	.16067	.15723	.14910	.13867	.13083	.11575	.10917	.10384	.10746
71	.22497	.18265	.16946	.15964	.15287	.13608	.12778	.12236	.11557	.10852	.10611	.10954
72	.22991	.17892	.16631	.15719	.15415	.14579	.11861	.11360	.10728	.09676	.09377	.09688
73	.22991	.18859	.17498	.15697	.15203	.13543	.12687	.12108	.11373	.10734	.10510	.10898
74	.22991	.18859	.17448	.16416	.16014	.15206	.14256	.13642	.12868	.12121	.11679	.12131

Table 4: Estimated Heteroschedastic error variance of Models

Observation	Model 1	Model 2	Model 3	Model 4	Model 5	Model 6	Model 7	Model 8	Model 9	Model 10	Model 11	Model 12
75	.22991	.18859	.17448	.16416	.16014	.15206	.14256	.13642	.12868	.12075	.11799	.12180
76	.22497	.18265	.16946	.15964	.15287	.13608	.12778	.12236	.11557	.10852	.10611	.10954
77	.22991	.17892	.16631	.15719	.15415	.14579	.11861	.11360	.10728	.09676	.09377	.09688
78	.22991	.18859	.17498	.15697	.15203	.13543	.12687	.12108	.11373	.10734	.10510	.10898
79	.22991	.18859	.17448	.16416	.16014	.15206	.14256	.13642	.12868	.12121	.11679	.12131
80	.22991	.18859	.17448	.16416	.16014	.15206	.14256	.13642	.12868	.12075	.11799	.12180
81	.22991	.18710	.17356	.16250	.15856	.14947	.14045	.13270	.12466	.11668	.11330	.11733
82	.22981	.18738	.17375	.16419	.16097	.15286	.14377	.13761	.12879	.12138	.11882	.12268
83	.22981	.18827	.17412	.16334	.16021	.15213	.14287	.13564	.12747	.11915	.11544	.12034
84	.22467	.18347	.16254	.15310	.14459	.13776	.12918	.12284	.11588	.10883	.10310	.11838
85	.22981	.18827	.17412	.16334	.16021	.15213	.14287	.13564	.12747	.11915	.11544	.12034
86	.22981	.18738	.17375	.16419	.15932	.14854	.13961	.13369	.12580	.11851	.11585	.11958
87	.22991	.18710	.17356	.15224	.14452	.13708	.12866	.12180	.10843	.10231	.10018	.10680
88	.22991	.18859	.17498	.16501	.16016	.15246	.14339	.13590	.12749	.12019	.11472	.11976
89	.22467	.18378	.17062	.16067	.15723	.14949	.14017	.13315	.12537	.11624	.11357	.11731
90	.22981	.18738	.17375	.16419	.16097	.15286	.14121	.13309	.12486	.11776	.11531	.11949
91	.22497	.18265	.16946	.15964	.15287	.13608	.12778	.12236	.11557	.10852	.10611	.10954
92	.21449	.17538	.16245	.15377	.15003	.12835	.12010	.11501	.10733	.10130	.09919	.10338
93	.22991	.18859	.16825	.15898	.15593	.14812	.13929	.12641	.11932	.11243	.10695	.11095
94	.22991	.18859	.17498	.16501	.16176	.15371	.14457	.13702	.12796	.12075	.11634	.12241
95	.22991	.18859	.16825	.15898	.15593	.14812	.13929	.12641	.11932	.11243	.10695	.11095
96	.22467	.18378	.17062	.16067	.15723	.14949	.14017	.13315	.12537	.11624	.11357	.11731
97	.22981	.18738	.17375	.16419	.16097	.15289	.14371	.13761	.12922	.12166	.11909	.12298
98	.22497	.18265	.16946	.15964	.15498	.14740	.13859	.13273	.12492	.11731	.11484	.11877
99	.22991	.18859	.17448	.16416	.16058	.15289	.14314	.13684	.12916	.12025	.11745	.12150
100	.22991	.18859	.17448	.16416	.16014	.15206	.14256	.13642	.12868	.12075	.11799	.12180
101	.22981	.18738	.16598	.15603	.15129	.14400	.13465	.12850	.12127	.11186	.10879	.11968
102	.22991	.18859	.17498	.16501	.16176	.15371	.14457	.13702	.12796	.12075	.11634	.12241

Table 5: Variable Weights for Observations

Observation	Model 1	Model 2	Model 3	Model 4	Model 5	Model 6	Model 7	Model 8	Model 9	Model 10	Model 11	Model 12
1	.34194	.29091	.27165	.25698	.25215	.23915	.20747	.19422	.17957	.16441	.15903	.16756
2	.34194	.29091	.27165	.25551	.25067	.23687	.22043	.20752	.19326	.17848	.17143	.17948
3	.33629	.28564	.25537	.24040	.22610	.21400	.19793	.18536	.17077	.15507	.14156	.17611
4	.34205	.29252	.27308	.25784	.25165	.23870	.22158	.20705	.19266	.17812	.17193	.18038
5	.34205	.27937	.25898	.24256	.23592	.22349	.20450	.19337	.17911	.15503	.14451	.15471
6	.34194	.29211	.27347	.25833	.25044	.23655	.22115	.20810	.19259	.17740	.17068	.17866
7	.33662	.28733	.26797	.25212	.24698	.23424	.21791	.20705	.19270	.17731	.17102	.17899
8	.34194	.29091	.27204	.25789	.25036	.23630	.22097	.20987	.19426	.17247	.15408	.16716
9	.33662	.28452	.26579	.25086	.24586	.22699	.20884	.19799	.18367	.15423	.14674	.15633
10	.33629	.28564	.26645	.24956	.24419	.23198	.21507	.20237	.17994	.16293	.15514	.16563
11	.34194	.29211	.26384	.23687	.23151	.21746	.20165	.18227	.16124	.14617	.13951	.15149
12	.34194	.29091	.27204	.25789	.23954	.22373	.20825	.19692	.18135	.16611	.12806	.14544
13	.34205	.29055	.27118	.25605	.24932	.23549	.22006	.20878	.19417	.17802	.17008	.18179
14	.34205	.29252	.27308	.25784	.25232	.24006	.22338	.20640	.18666	.16835	.16299	.17392
15	.34205	.29252	.27308	.25784	.25165	.23870	.22158	.21006	.19559	.16285	.15743	.16591
16	.34194	.29091	.26060	.24515	.23744	.22509	.20830	.19662	.18213	.16194	.15498	.17884
17	.34205	.29055	.27118	.25605	.24932	.23549	.22006	.20878	.15438	.13080	.12537	.13332
18	.34205	.29055	.27176	.25727	.24354	.23139	.21583	.20503	.18933	.17352	.16815	.17682
19	.32470	.27398	.25406	.23996	.23401	.22121	.20505	.19341	.17698	.16247	.15458	.16262
20	.34205	.29252	.27308	.25784	.24495	.23293	.21548	.20466	.18520	.16777	.16238	.17388
21	.34205	.29252	.27308	.25784	.25232	.24006	.22338	.21250	.19341	.17545	.16914	.17853
22	.34205	.29252	.26399	.24982	.24499	.23213	.21676	.19252	.17808	.16321	.15072	.15990
23	.34194	.29211	.27256	.25779	.24806	.23295	.21491	.20409	.18959	.17357	.16769	.17605
24	.34205	.29055	.25992	.24385	.23354	.22131	.20425	.19290	.16937	.15381	.14479	.16497
25	.34205	.27937	.26110	.24700	.24211	.22980	.21384	.20190	.18683	.16848	.15565	.16741
26	.34205	.29252	.27380	.25913	.25416	.24139	.22606	.21266	.19556	.18106	.17175	.18448
27	.34194	.29211	.27256	.25779	.25291	.24052	.22405	.21301	.19806	.18305	.17658	.18453
28	.32470	.27437	.25522	.24149	.23534	.19633	.17971	.16889	.15161	.13714	.13188	.14223
29	.34205	.29055	.27176	.25530	.24916	.23708	.22117	.14580	.13010	.11465	.10446	.11294
30	.34194	.29091	.27204	.25789	.25294	.24001	.22469	.21375	.19717	.18236	.17704	.18502
31	.34205	.29252	.27380	.25913	.24870	.23444	.21189	.20024	.18428	.16975	.16260	.17053
32	.34205	.29252	.27308	.25784	.25165	.23870	.22257	.21157	.19697	.18201	.17272	.18223
33	.34205	.29252	.27380	.25913	.25416	.24139	.22168	.20937	.19177	.17615	.16643	.18107
34	.34205	.29252	.27380	.24664	.23866	.20974	.17979	.16034	.14493	.12697	.12153	.12976
35	.34205	.29252	.26399	.24982	.24097	.22868	.21331	.19849	.17866	.16188	.11895	.12983
36	.34205	.29055	.27118	.25564	.25055	.23706	.22053	.20740	.19301	.17271	.16529	.17324
37	.33629	.28607	.26749	.25247	.24706	.23378	.21811	.20499	.18901	.17439	.16829	.17623

Table 6: Variable Weights for Observations

Observation	Model 1	Model 2	Model 3	Model 4	Model 5	Model 6	Model 7	Model 8	Model 9	Model 10	Model 11	Model 12
38	.34205	.29252	.27308	.25770	.25123	.23752	.22011	.20821	.19294	.17690	.16988	.17784
39	.32470	.27398	.25406	.23996	.23410	.22192	.20517	.18985	.17386	.15902	.15121	.15927
40	.34205	.29055	.27176	.25530	.24916	.23440	.21885	.20466	.18903	.17249	.16515	.17387
41	.34205	.29055	.27176	.25727	.25139	.23767	.22236	.21144	.19507	.17464	.16911	.17832
42	.33629	.28564	.26645	.25214	.23541	.22025	.20384	.18900	.17017	.13879	.11768	.12582
43	.34205	.29055	.27176	.25530	.24217	.22964	.21323	.20203	.18643	.16828	.16262	.17351
44	.33629	.28607	.26636	.25197	.24697	.23440	.20614	.19355	.17506	.16001	.15435	.16659
45	.34194	.29211	.27256	.25779	.24806	.23295	.21724	.20142	.18059	.16493	.15240	.17354
46	.34194	.29211	.27256	.25659	.25175	.23882	.22311	.21012	.19461	.17772	.16982	.18022
47	.34205	.29252	.27380	.25771	.23215	.21992	.19830	.18616	.17068	.14487	.13810	.14691
48	.34194	.29091	.27204	.25485	.24996	.23759	.22139	.20991	.19297	.17796	.17193	.17987
49	.34205	.27937	.25898	.24435	.23917	.22510	.20930	.19596	.18162	.16708	.16041	.17332
50	.34205	.29252	.27308	.25770	.25235	.23965	.22141	.20983	.19465	.17928	.17308	.18107
51	.34205	.29055	.27176	.25530	.24916	.23708	.21955	.20734	.19086	.17048	.16408	.17327
52	.34205	.29252	.27380	.25771	.24670	.23467	.21591	.20489	.18867	.16991	.16131	.17077
53	.34205	.29055	.27176	.23899	.22599	.21276	.19692	.18323	.15415	.13964	.13438	.15037
54	.34205	.29252	.27308	.25784	.25232	.24006	.22359	.21233	.19790	.18344	.17810	.18677
55	.33662	.28452	.26501	.24929	.24288	.23028	.21445	.20354	.18591	.17043	.16429	.17692
56	.34205	.29055	.27118	.25605	.24932	.23549	.21649	.20448	.17765	.16196	.15522	.16354
57	.34205	.29252	.27308	.25770	.25235	.23965	.22141	.20703	.19205	.17654	.16915	.17988
58	.34205	.29252	.27308	.25784	.24495	.22934	.21094	.20014	.18585	.17120	.16594	.17489
59	.34194	.29091	.27204	.23738	.22763	.21412	.19643	.18339	.16893	.14914	.14383	.15340
60	.33662	.28733	.25829	.24008	.23465	.22040	.20509	.19320	.16173	.14333	.13759	.14798
61	.34205	.29252	.26399	.24672	.24187	.22952	.21020	.19600	.17676	.16155	.15625	.17848
62	.34194	.29211	.27256	.25659	.23617	.22031	.20375	.19082	.17445	.15898	.15317	.16267
63	.34194	.29091	.27204	.25485	.24690	.23170	.21282	.18480	.16756	.15256	.14558	.15388
64	.34194	.29091	.27204	.25789	.25294	.24001	.22019	.20538	.18943	.17480	.16953	.17843
65	.33629	.28564	.26665	.25275	.24486	.23247	.19836	.18489	.16844	.14756	.14220	.15238
66	.33662	.28452	.26579	.25086	.24586	.22699	.20884	.19799	.18367	.15423	.14674	.15633
67	.33662	.28452	.26501	.24929	.24288	.23028	.19961	.18583	.16929	.14635	.13976	.14914
68	.34194	.29091	.27204	.25485	.24996	.23626	.22050	.20873	.19065	.17429	.16892	.17686
69	.33629	.28607	.26749	.25247	.24706	.23444	.21835	.20549	.19045	.17155	.16574	.17383
70	.33629	.28607	.26749	.25247	.24706	.23378	.21565	.20110	.17049	.15585	.14333	.15191
71	.33662	.28452	.26579	.25086	.24003	.21095	.19521	.18438	.17011	.15436	.14875	.15671
72	.34205	.27937	.26110	.24700	.24211	.22817	.17659	.16580	.15150	.12569	.11785	.12599
73	.34205	.29252	.27380	.24664	.23866	.20974	.19341	.18175	.16609	.15164	.14637	.15541
74	.34205	.29252	.27308	.25784	.25165	.23870	.22257	.21157	.19697	.18201	.17272	.18223

Table 7: Variable Weights for Observations

Observation	Model 1	Model 2	Model 3	Model 4	Model 5	Model 6	Model 7	Model 8	Model 9	Model 10	Model 11	Model 12
75	.34205	.29252	.27308	.25784	.25165	.23870	.22257	.21157	.19697	.18106	.17529	.18323
76	.33662	.28452	.26579	.25086	.24003	.21095	.19521	.18438	.17011	.15436	.14875	.15671
77	.34205	.27937	.26110	.24700	.24211	.22817	.17659	.16580	.15150	.12569	.11785	.12599
78	.34205	.29252	.27380	.24664	.23866	.20974	.19341	.18175	.16609	.15164	.14637	.15541
79	.34205	.29252	.27308	.25784	.25165	.23870	.22257	.21157	.19697	.18201	.17272	.18223
80	.34205	.29252	.27308	.25784	.25165	.23870	.22257	.21157	.19697	.18106	.17529	.18323
81	.34205	.29055	.27176	.25530	.24916	.23440	.21885	.20466	.18903	.17249	.16515	.17387
82	.34194	.29091	.27204	.25789	.25294	.24001	.22469	.21375	.19717	.18236	.17704	.18502
83	.34194	.29211	.27256	.25659	.25175	.23882	.22311	.21012	.19461	.17772	.16982	.18022
84	.33629	.28564	.25537	.24040	.22610	.21400	.19793	.18536	.17077	.15507	.14156	.17611
85	.34194	.29211	.27256	.25659	.25175	.23882	.22311	.21012	.19461	.17772	.16982	.18022
86	.34194	.29091	.27204	.25789	.25036	.23284	.21735	.20650	.19131	.17639	.17070	.17864
87	.34205	.29055	.27176	.23899	.22599	.21276	.19692	.18323	.15415	.13964	.13438	.15037
88	.34205	.29252	.27380	.25913	.25167	.23935	.22402	.21061	.19463	.17991	.16826	.17901
89	.33629	.28607	.26749	.25247	.24706	.23444	.21835	.20549	.19045	.17155	.16574	.17383
90	.34194	.29091	.27204	.25789	.25294	.24001	.22019	.20538	.18943	.17480	.16953	.17843
91	.33662	.28452	.26579	.25086	.24003	.21095	.19521	.18438	.17011	.15436	.14875	.15671
92	.32470	.27437	.25522	.24149	.23534	.19633	.17971	.16889	.15161	.13714	.13188	.14223
93	.34205	.29252	.26399	.24982	.24499	.23213	.21676	.19252	.17808	.16321	.15072	.15990
94	.34205	.29252	.27380	.25913	.25416	.24139	.22606	.21266	.19556	.18106	.17175	.18448
95	.34205	.29252	.26399	.24982	.24499	.23213	.21676	.19252	.17808	.16321	.15072	.15990
96	.33629	.28607	.26749	.25247	.24706	.23444	.21835	.20549	.19045	.17155	.16574	.17383
97	.34194	.29091	.27204	.25789	.25294	.24005	.22458	.21373	.19800	.18294	.17760	.18563
98	.33662	.28452	.26579	.25086	.24346	.23092	.21551	.20471	.18954	.17383	.16852	.17692
99	.34205	.29252	.27308	.25784	.25232	.24006	.22359	.21233	.19790	.18003	.17413	.18261
100	.34205	.29252	.27308	.25784	.25165	.23870	.22257	.21157	.19697	.18106	.17529	.18323
101	.34194	.29091	.26060	.24515	.23744	.22509	.20830	.19662	.18213	.16194	.15498	.17884
102	.34205	.29252	.27380	.25913	.25416	.24139	.22606	.21266	.19556	.18106	.17175	.18448

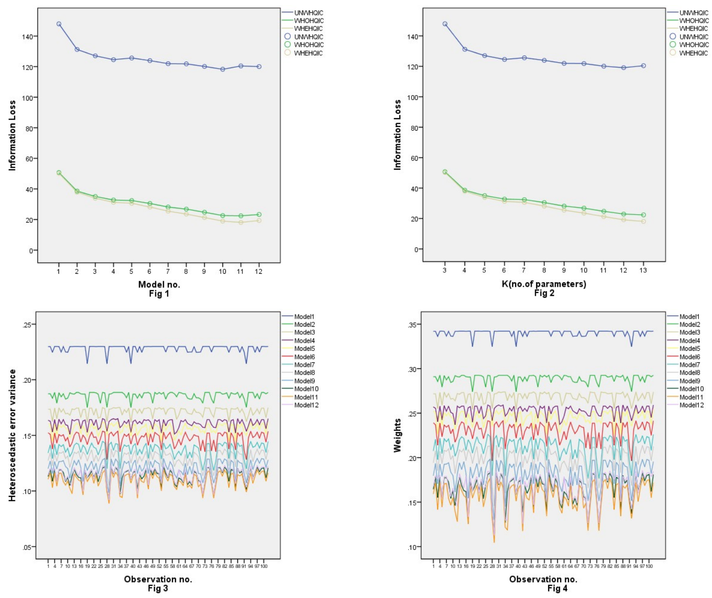

Figure 1: Lineplot shows the information loss of Models based on no.of parameters and Heteoscedastic error variance, Weights for observations

Table-1 exhibits the result of the stepwise regression analysis, the traditional unweighted Schwarz Bayesian information criterion and weighted Schwarz Bayesian Information criterion under two versions for the 12 fitted nested models. From the results, the authors found model 11 is having minimum homoschedastic error variance 0f 0.123 with a high R2 of 63.4%, but the unweighted Schwarz Bayesian information criterion is found to be a minimum of 155.969 for Model-2. This shows model 2 is better when compared to other competing models. But this selection is biased, because this model is having poor fitness when compared it with others. As far as, the proposed homogeneous weighted SBIC for Model 10 is penalized by a homogeneous weight of 0.191 and we get the value of homogeneous SBIC as 36.769 which is minimum when compared to other competing models. On the other hand, the heterogeneous weighted SBIC assumed that the point wise information loss should not be equally weighted and it should weighed with variable weights. The heterogeneous WSBIC for Model 11 is minimum (30.710) when it is compared with other fitted regression models. This resembles the homogeneous and heterogeneous weighted SBIC gives dissimilar results and it is different from the results given by unweighted traditional SBIC. If the error variances of the fitted models are heteroschedastic, using the UNWSBIC for model selection is impractical. Hence, the application of homogeneous and heterogeneous WSBIC helps the decision maker to select and finalize the best model as model 10 as per homogeneous weighted SBIC and model-11 as per heterogeneous weighted SBIC. Another important feature of the two versions of WSBIC is R2 supportive selection and the penalization of the model was balanced by the estimated weights proposed by authors. Finally the authors emphasize, if the heteroschedasticity is existing in the survey data then using the weighted SBIC will give an appropriate and alternative selection of models among a set of competing models. The subsequent tables and line plots exhibit the estimated heteroschedastic error variance of 12 fitted models and the extracted variable weights for 102 observations.

5 Conclusion

This paper proposed new information criteria as Weighted Schwarz Bayesian information criterion which is an alternative to the traditional Schwarz Bayesian information criterion existing in the literature. The proposed WSBIC is superior in two different aspects. At first the weighted Schwarz Bayesian information criterion incorporates the heteroschedastic error variance of the fitted models and secondly it gives unequal weights to the point wise information loss to the fitted models. The authors emphasize, if the problem of heteroschedasticity is present in the data, the usage of traditional Schwarz Bayesian information criterion for model selection will leads the researchers to select wrong model. Because the traditional Schwarz Bayesian information criterion works perfectly when the error variance of the fitted model is homoschedastic and this assumption is violated, the application of alternative information criteria under two different versions namely Homogeneous and Heterogeneous WSBIC was proposed by the authors. For future research, the authors recommended that the derivation can be extended to the logical extraction of log likelihood based information criteria.

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