Electronic Journal of Applied Statistical Analysis
EJASA, Electron. J. App. Stat. Anal.
http://siba-ese.unisalento.it/index.php/ejasa/index
e-ISSN: 2070-5948
DOI: 10.1285/i20705948v7n1p94

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April 26, 2014

Electronic Journal of Applied Statistical Analysis
Vol. 7, Issue 1, 2014, 94-116
DOI: 10.1285/i20705948v7n1p94

Generalizations of the inverse Weibull and related distributions with application

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April 26, 2014

In this paper, the generalized inverse Weibull distribution including the exponentiated or proportional reverse hazard and Kumaraswamy generalized inverse Weibull distributions are presented. Properties of these distributions including the behavior of the hazard and reverse hazard functions, moments, coefficients of variation, skewness, and kurtosis, $\beta-$entropy, Fisher information matrix are studied. Estimates of the model parameters via method of maximum likelihood (ML), and method of moments (MOM) are presented for complete and censored data. Numerical examples are also presented.

keywords: Inverse Weibull Distribution, Proportional Inverse Weibull Distribution, Generalized Distribution.

1 Introduction

The inverse Weibull distribution can be readily applied to a wide range of situations including applications in medicine, reliability and ecology. Keller et al. (1985) obtained the inverse Weibull model by investigating failures of mechanical components subject to degradation. Calabria and Pulcini (1990) computed the maximum likelihood and least squares estimates of the parameters of the inverse Weibull distribution. They also obtained the Bayes estimator of the model parameters as well as confidence limits for reliability and tolerance limits. See Calabria and Pulcini (1989, 1994), and Johnson et al. (1984) for additional details. Khan et al. (2008) presented some important theoretical properties of the inverse Weibull distribution. Samanta and

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ISSN: 2070-5948
http://siba-ese.unisalento.it/index.php/ejasa/index

Bhowmick (2010) presented a deterministic inventory system with Weibull distribution deterioration and ramp type demand rate. The inverse Weibull (IW) cumulative distribution function (cdf) is given by

$$F(x;\alpha,\beta) = exp\left[-(\alpha(x-x_0))^{-\beta}\right], \qquad x \geq x_0,\, \alpha > 0,\, \beta > 0, \tag{1}$$

where α, x_0 and β are the scale, location and shape parameters, respectively. Often the parameter x_0 is called the minimum life or guarantee time. When $\alpha = 1$ and $x = x_0 + \alpha$, then $F(\alpha + x_0; 1; \beta) = F(\alpha + x_0; 1) = e^{-1} = 0.3679$. This value is in fact the characteristic life of the distribution. In what follows, we assume that $x_0 = 0$, and the inverse Weibull cdf becomes

$$F(x;\alpha,\beta) = exp[-(\alpha x)^{-\beta}], \qquad x \geq 0,\, \alpha > 0,\, \beta > 0. \tag{2}$$

Note that when $\alpha = 1$, we have the Fréchet distribution function. The inverse Weibull probability density function (pdf) is given by

$$f(x;\alpha,\beta) = \beta\alpha^{-\beta}x^{-\beta-1}exp[-(\alpha x)^{-\beta}], \qquad x \geq 0,\, \alpha > 0,\, \beta > 0. \tag{3}$$

When $\beta = 1$ and $\beta = 2$, the inverse Weibull distribution pdfs are referred to as the inverse exponential and inverse Raleigh pdfs, respectively. The k^{th} raw or non central moments are given by

$$E(X^k) = \frac{\Gamma(1-k/\beta)}{\alpha^k}, \quad \text{for} \quad \beta > k. \tag{4}$$

Note that $E(X^k)$ does not exist when $\beta \leq k$. See Johnson et al. (1984).

This paper is organized as follows. Section 2 contains some basic utility notions. In section 3, the exponentiated or proportional inverse Weibull (PIW) and Kumaraswamy generalized inverse Weibull distributions are presented. The mode, hazard function and reverse hazard function are also presented in section 3. Glaser's Lemma is applied to the PIW distribution to determine the behavior of the hazard function. In section 4, the moments, entropy measures and Fisher information are presented. Estimation of the parameters of the PIW distribution via the methods of moments and maximum likelihood as well as numerical examples for complete and right censored data are presented in section 5. Section 6 deals with Kumaraswamy generalized inverse Weibull distribution. The mode, hazard function, reverse hazard function, moments, Shannon entropy and estimates of the model parameters for censored data are presented.

2 Basic Utility Notions

In this section, some basic utility notions and definitions are presented. Suppose the distribution of a continuous random variable X has the parameter set $\theta^* = \{\theta_1, \theta_2, \cdots, \theta_n\}$. Let the pdf of the random variable X be given by $f(x;\theta^*)$. The hazard function of X can be interpreted as the instantaneous failure rate or the conditional probability density of failure at time x, given that the unit has survived until time x, see Shaked and Shanthikumar (1994). The hazard function $h(x;\theta^*)$ is defined to be

$$h(x;\theta^*) = \lim_{\Delta x \to 0} \frac{P(x \leq X \leq x+\Delta x)}{\Delta x[1-F(x;\theta^*)]} = \frac{-\bar{F}'(x;\theta^*)}{\bar{F}(x;\theta^*)} = \frac{f(x;\theta^*)}{1-F(x;\theta^*)}, \tag{5}$$

where $\bar{F}(x;\theta^*)$ is the survival or reliability function. The concept of reverse hazard rate was introduced as the hazard rate in the negative direction and received minimal attention, if any, in the literature. Keilson and Sumita (1982) demonstrated the importance of the reverse hazard rate and reverse hazard orderings. Shaked and Shanthikumar (1994) presented results on reverse hazard rate. See Ross (1983), Chandra and Roy (2001), Block and Savits (1998) for additional details. We present a formal definition of the reverse hazard function of a distribution function F. The reverse Hazard function can be interpreted as an approximate probability of a failure in $[x, x+dx]$, given that the failure had occurred in $[0,x]$.

Definition 2.1. *Let (a, b), $-\infty \leq a < b < \infty$, be an interval of support for F. Then the reverse hazard function of X (or F) at $t > a$ is denoted by $\tau_F(t)$ and is defined as*

$$\tau(t;\theta^*) = \frac{d}{dt}\log F(t;\theta^*) = \frac{f(t;\theta^*)}{F(t;\theta^*)}. \tag{6}$$

Some useful functions that are employed in subsequent sections are given below. The gamma and digamma functions are given by $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt$ and $\Psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ respectively, where $\Gamma'(x) = \int_0^\infty t^{x-1}(\log t)e^{-t}\,dt$ is the first derivative of the gamma function.

Definition 2.2. *The n^{th}-order derivative formula of gamma function is given by:*

$$\Gamma^{(n)}(s) = \int_0^\infty z^{s-1}(\log z)^n \exp(-z)\,dz. \tag{7}$$

This derivative will be used frequently in this paper. The lower incomplete gamma function and the upper incomplete gamma function are

$$\gamma(s,x) = \int_0^x t^{s-1}e^{-t}\,dt \quad \text{and} \quad \Gamma(s,x) = \int_x^\infty t^{s-1}e^{-t}\,dt, \tag{8}$$

respectively.

3 Generalized Inverse Weibull Distributions

The proportional inverse Weibull (PIW) distribution has a cdf given by

$$G(x;\alpha,\beta,\gamma) = [F(x)]^\gamma = \exp[-\gamma(\alpha x)^{-\beta}], \quad \text{for } \alpha > 0,\ \beta > 0,\ \gamma > 0, \text{ and } x \geq 0. \tag{9}$$

The corresponding pdf is given by

$$g(x;\alpha,\beta,\gamma) = \alpha\beta\gamma(\alpha x)^{-\beta-1}\exp[-\gamma(\alpha x)^{-\beta}], \tag{10}$$

for $\alpha > 0$, $\beta > 0$, $\gamma > 0$, and $x \geq 0$.

Jones (2009) explored the background and genesis of the Kumaraswamy distribution (Kumaraswamy (1980)) and, more importantly, made clear some similarities and differences between the beta and Kumaraswamy distributions. Among the advantages are: the normalizing constant is very simple; the distribution and quantile functions have simple explicit formula which do not involve special functions; explicit formula for moments of order statistics and L-moments. However, compared to Kumaraswamy distribution, the beta distribution has the following advantages: simpler formula for moments and moment generating function (mgf); a one-parameter sub-family of symmetric distributions; simpler moment estimation and more ways of generating the distribution via physical processes. Cordeiro and Ortega (2010) among others studied the Kumaraswamy Weibull distribution and applied it to failure time data.

Kumaraswamy (1980) in his paper proposed a two-parameter distribution (Kumaraswamy distribution) defined in (0, 1). Here we will refer to it as Kum distribution. Its cdf is given by:

$$F(x;a;b) = 1-(1-x^a)^b, \quad x \in (0,1), a > 0, b > 0. \tag{11}$$

The parameters a and b are the shape parameters. The Kum distribution has the pdf given by:

$$f(x;a,b) = abx^{a-1}(1-x^a)^{b-1}, \quad x \in (0,1), a > 0, b > 0. \tag{12}$$

The Kum-Generalized Inverse Weibull (KGIW) cdf is given by

$$\begin{aligned} G(x;\alpha,\beta,\lambda,\varphi) &= 1-(1-F^{\lambda}(x;\alpha,\beta))^{\varphi} \\ &= 1-\{1-\exp[-\lambda(\alpha x)^{-\beta}]\}^{\varphi}, \end{aligned}$$

for $x > 0$, $\alpha > 0$, $\beta > 0$, $\lambda > 0$, and $\varphi > 0$. The corresponding KGIW pdf is given by

$$g(x;\alpha,\beta,\lambda,\varphi) = \alpha\beta\lambda\varphi(\alpha x)^{-\beta-1}\exp[-\lambda(\alpha x)^{-\beta}]\{1-\exp[-\lambda(\alpha x)^{-\beta}]\}^{\varphi-1}$$

for $x > 0$, $\alpha > 0$, $\beta > 0$, $\lambda > 0$, and $\varphi > 0$.

3.1 Mode of the Proportional Inverse Weibull Distribution

Consider the PIW distribution. Note that,

$$\ln g(x;\alpha,\beta,\gamma) = \ln(\alpha\beta\gamma) - (1+\beta)\ln(\alpha x) - \gamma(\alpha x)^{-\beta}.$$

Differentiating $\ln g(x;\alpha,\beta,\gamma)$ with respect to x, we obtain

$$\begin{aligned} \frac{\partial \ln g(x;\alpha,\beta,\gamma)}{\partial x} &= -\frac{1+\beta}{x} + \alpha\beta\gamma(\alpha x)^{-\beta-1} \\ &= \frac{1}{x}\left[\frac{\alpha^{-\beta}\beta\gamma}{x^{\beta}} - (1+\beta)\right]. \end{aligned}$$

Now, set $\frac{\partial \ln g(x;\alpha,\beta,\gamma)}{\partial x}$ equal 0 and solve for x, to get

$$x_0 = \left(\frac{\alpha^{-\beta}\beta\gamma}{1+\beta}\right)^{\frac{1}{\beta}}.$$

Obviously, when $0 < x < (\frac{\alpha^{-\beta}\beta\gamma}{1+\beta})^{\frac{1}{\beta}}, \frac{\partial \ln g(x;\alpha,\beta,\gamma)}{\partial x} > 0$, so $g(x;\alpha,\beta,\gamma)$ is increasing, and when $x > (\frac{\alpha^{-\beta}\beta\gamma}{1+\beta})^{\frac{1}{\beta}}$, $g(x;\alpha,\beta,\gamma)$ is decreasing, so $g(x;\alpha,\beta,\gamma)$ attains a maximum when $x_0 = (\frac{\alpha^{-\beta}\beta\gamma}{1+\beta})^{\frac{1}{\beta}}$. Note that x_0 is the mode of PIW distribution.

3.2 Hazard Function

The hazard function of the PIW distribution is given by

$$\lambda_G(x;\alpha,\beta,\gamma) = \frac{g(x;\alpha,\beta,\gamma)}{\bar{G}(x;\alpha,\beta,\gamma)} = \frac{\alpha\beta\gamma(\alpha x)^{-\beta-1}exp[-\gamma(\alpha x)^{-\beta}]}{1-exp[-\gamma(\alpha x)^{-\beta}]},$$

for $\gamma > 0$, $\alpha > 0$, $\beta > 0$, $x \geq 0$.

We study the behavior of the hazard function of the PIW distributions via Glaser (1980) lemma . Note that

$$\begin{aligned} \eta_G(x;\alpha,\beta,\gamma) &= \eta_G(x) \\ &= -\frac{g'(x;\alpha,\beta,\gamma)}{g(x;\alpha,\beta,\gamma)} \\ &= \alpha(1+\beta)(\alpha x)^{-1} - \alpha\beta\gamma(\alpha x)^{-\beta-1}, \end{aligned}$$

and

$$\eta'_G(x) = \alpha^2\beta\gamma(\beta+1)(\alpha x)^{-\beta-2} - \alpha^2(1+\beta)(\alpha x)^{-2}.$$

Let $\eta'_G(x) = 0$, we get $x_0 = \frac{1}{\alpha}(\beta\gamma)^{\frac{1}{\beta}}$. So when $0 < x < x_0$, $\eta'_G(x) > 0$, $\eta'_G(x_0) = 0$ and when $x > x_0$, $\eta'_G(x) < 0$. So the hazard function is *upside down bathtub shape*.

3.3 Reverse Hazard Function

The reverse hazard function for the PIW distribution is given by

$$\tau_G(x;\alpha,\beta,\gamma) = \frac{g(x;\alpha,\beta,\gamma)}{G(x;\alpha,\beta,\gamma)} = \alpha\beta\gamma(\alpha x)^{-\beta-1},$$

for $\gamma > 0$, $\alpha > 0$, $\beta > 0$, $x \geq 0$.

4 Moments, Entropy and Fisher Information

In this section, we present the moments and related functions for the proportional inverse Weibull distribution. The concept of entropy plays a vital role in information theory. The entropy of a random variable is defined in terms of its probability distribution and can be shown to be a good measure of randomness or uncertainty. We present Shannon entropy and $\beta-$entropy for the PIW distribution. Also, presented is Fisher information matrix for the PIW distribution.

4.1 Moments

The moments of the PIW distribution are given by

$$\begin{aligned} E(X^c) &= \int_0^\infty x^c \cdot g(x;\alpha,\beta,\gamma)\,dx \\ &= \int_0^\infty \alpha\beta\gamma \cdot x^c \cdot (\alpha x)^{-\beta-1} exp[-\gamma(\alpha x)^{-\beta}]\,dx \\ &= \gamma^{\frac{c}{\beta}} \cdot \alpha^{-c}\Gamma\left(\frac{\beta-c}{\beta}\right), \end{aligned}$$

where $\beta > c$. The variance is given by

$$\sigma^2 = E(X^2) - E^2(X) = \gamma^{\frac{2}{\beta}}\alpha^{-2}\Big[\Gamma\Big(\frac{\beta-2}{\beta}\Big) - \Gamma^2\Big(\frac{\beta-1}{\beta}\Big)\Big].$$

The coefficient of variation (CV) is given by

$$\begin{aligned} CV = \frac{\sigma}{\mu} &= \frac{\gamma^{\frac{1}{\beta}}\alpha^{-1}\sqrt{\Gamma(\frac{\beta-2}{\beta}) - \Gamma^2(\frac{\beta-1}{\beta})}}{\gamma^{\frac{1}{\beta}}\alpha^{-1}\Gamma(\frac{\beta-1}{\beta})} \\ &= \frac{\sqrt{\Gamma(\frac{\beta-2}{\beta}) - \Gamma^2(\frac{\beta-1}{\beta})}}{\Gamma(\frac{\beta-1}{\beta})}. \end{aligned}$$

The coefficient of Skewness (CS) is given by

$$CS = \frac{2\Gamma^3(\frac{\beta-1}{\beta}) - 3\Gamma(\frac{\beta-1}{\beta})\Gamma(\frac{\beta-2}{\beta}) + \Gamma(\frac{\beta-3}{\beta})}{[\Gamma(\frac{\beta-2}{\beta}) - \Gamma^2(\frac{\beta-1}{\beta})]^{\frac{3}{2}}}.$$

The coefficient of Kurtosis (CK) is given by

$$CK = \frac{\Gamma(\frac{\beta-4}{\beta}) - 4\Gamma(\frac{\beta-1}{\beta})\Gamma(\frac{\beta-3}{\beta}) + 6\Gamma^2(\frac{\beta-1}{\beta})\Gamma(\frac{\beta-2}{\beta}) - 3\Gamma^4(\frac{\beta-1}{\beta})}{[\Gamma(\frac{\beta-2}{\beta}) - \Gamma^2(\frac{\beta-1}{\beta})]^2}.$$

The graphs of the mean against β for values of α and γ shows a decreasing mean for increasing values of β. Table 1 shows the mode, mean, standard deviation (STD), CV, CS and CK for some values of the parameters α, β and γ. From the table 1, we can see that as β increases, the Mean, STD, coefficient of variation, skewness and kurtosis are all decreasing. Graph of the mean against β for some values of the parameters α, and γ shows a decreasing trend.

4.2 Shannon Entropy

Shannon entropy for PIW distribution is given by

$$\begin{aligned} H(g) &= E[-\log g(X)] \\ &= -\int_0^\infty [\log g(x)]g(x)\,dx \\ &= -\alpha\beta\gamma[A+B+C], \end{aligned}$$

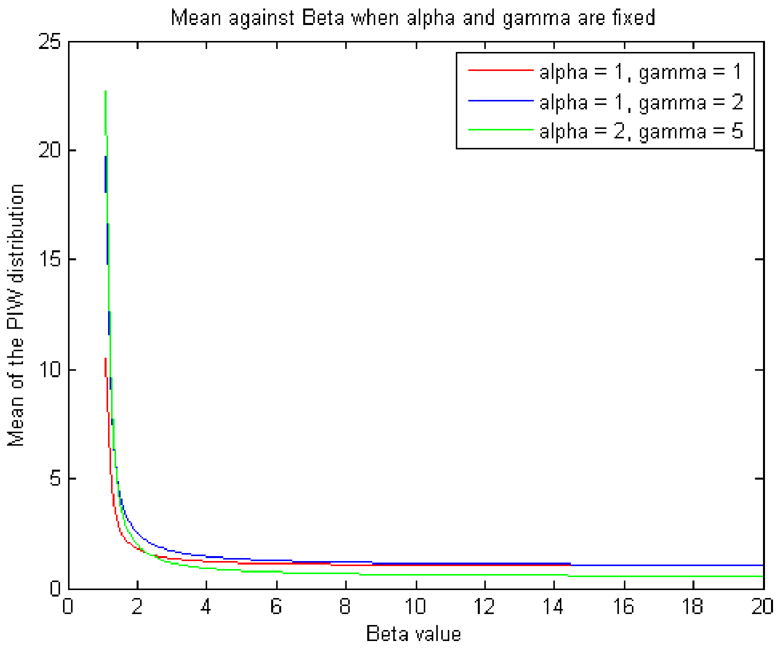

Figure 1: **Mean of PIW Distribution**

Table 1: **Mode, Mean, STD, Coefficients of Variation, Skewness and Kurtosis**

α	β	γ	Mode	Mean	STD	CV	CS	CK
1	5	1	0.9641925	1.164230	0.3657341	0.3141425	3.535072	48.09151
1	6	2	1.093991	1.267021	0.3173964	0.2505061	2.805566	24.67812
1	8	3	1.130436	1.250052	0.2238499	0.1790725	2.189270	14.16589
1	11	4	1.125375	1.203964	0.1514753	0.1258139	1.820604	10.10796
2	12	5	0.5679638	0.6035245	0.06910668	0.1145052	1.749809	9.468404
3	13	6	0.3804178	0.4019915	0.04223783	0.1050714	1.692484	8.979469
4	14	7	0.2858669	0.3006762	0.0291897	0.09708017	1.645094	8.593978
5	15	8	0.2287533	0.2396567	0.02162252	0.09022286	1.605245	8.282494

where A, B and C are obtained below:

$$\begin{aligned} A &= \int_0^\infty \log(\alpha\beta\gamma)(\alpha x)^{-\beta-1}\exp[-\gamma(\alpha x)^{-\beta}]\,dx \\ &= \frac{\log(\alpha\beta\gamma)}{\alpha\beta\gamma}, \end{aligned}$$

$$
\begin{aligned}
B &= -(1+\beta)\int_0^\infty \log(\alpha x)(\alpha x)^{-\beta-1}\exp[-\gamma(\alpha x)^{-\beta}]\,dx \\
&= \frac{1+\beta}{\alpha\beta}\int_0^\infty \log(x)\exp[-\gamma x^{-\beta}]\,dx^{-\beta} \\
&= \frac{1+\beta}{\alpha\beta^2\gamma}\left[\int_0^\infty \log x \exp(-x)\,dx - \log\gamma\int_0^\infty \exp[-x]\,dx\right],
\end{aligned}
$$

using the fact that $\Gamma^n(t) = \int_0^\infty \log^n(x)x^{t-1}\exp(-x)\,dx$, we obtain

$$
B = -\frac{1+\beta}{\alpha\beta^2\gamma}\log\gamma,
$$

and

$$
\begin{aligned}
C &= -\frac{\gamma}{\alpha}\int_0^\infty (\alpha x)^{-2\beta-1}\exp[-\gamma(\alpha x)^{-\beta}]\,d\alpha x \\
&= -\frac{1}{\alpha\beta\gamma}.
\end{aligned}
$$

Finally, Shannon entropy reduces to

$$
H(g) = \frac{1}{\beta} + 2 - \log(\alpha\beta\gamma).
$$

4.3 β- Entropy

β-entropy is a one parameter generalization of the Shannon entropy. β- entropy is defined by

$$
H_{\tilde{\beta}}(g) = \frac{1}{\tilde{\beta}-1}\left[1 - \int_0^\infty g^{\tilde{\beta}}(x)\,dx\right], \quad \text{for } \tilde{\beta} \neq 1.
$$

β-entropy for PIW distribution is given by

$$
H_{\tilde{\beta}}(g) = \frac{1}{\tilde{\beta}-1}\left[1 - \int_0^\infty \alpha^{\tilde{\beta}}\beta^{\tilde{\beta}}\gamma^{\tilde{\beta}}(\alpha x)^{-(\beta+1)\tilde{\beta}}\exp[-\gamma\tilde{\beta}(\alpha x)^{-\beta}]\right] dx.
$$

Let $t = \gamma\tilde{\beta}(\alpha x)^{-\beta}$, so $x = \frac{1}{\alpha}\left(\frac{t}{\gamma\tilde{\beta}}\right)^{\frac{1}{\beta}}$, then we have

$$
\begin{aligned}
H_{\tilde{\beta}}(g) &= \frac{1}{\tilde{\beta}-1}\left[1 - \alpha^{\tilde{\beta}-1}\beta^{\tilde{\beta}-1}\gamma^{\tilde{\beta}-1}\int_0^\infty \frac{1}{\tilde{\beta}}\left(\frac{t}{\gamma\tilde{\beta}}\right)^{\frac{\tilde{\beta}\beta+\tilde{\beta}-\beta-1}{\beta}}\exp[-t]\,dt\right] \\
&= \frac{1}{\tilde{\beta}-1}\left[1 - \alpha^{\tilde{\beta}-1}\beta^{\tilde{\beta}-1}\gamma^{\frac{1-\tilde{\beta}}{\beta}}\tilde{\beta}^{\frac{1-\tilde{\beta}-\tilde{\beta}\beta}{\beta}}\int_0^\infty t^{\frac{\tilde{\beta}\beta+\tilde{\beta}-1}{\beta}-1}\exp[-t]\,dt\right] \\
&= \frac{1}{\tilde{\beta}-1}\left[1 - \alpha^{\tilde{\beta}-1}\beta^{\tilde{\beta}-1}\gamma^{\frac{1-\tilde{\beta}}{\beta}}\tilde{\beta}^{\frac{1-\tilde{\beta}-\tilde{\beta}\beta}{\beta}}\Gamma\left(\frac{\tilde{\beta}+\tilde{\beta}\beta-1}{\beta}\right)\right].
\end{aligned}
$$

4.4 Fisher Information

The information (or Fisher information) that a random variable X contains about the parameter θ is given by

$$I(\theta) = E\left[\frac{\partial}{\partial\theta} log(f(X,\theta))\right]^2.$$

Now, if $\log(f(X,\theta))$ is twice differentiable with respect to θ, and under certain regularity conditions (Lehmann (1998)), Fisher Information is given by

$$I(\theta) = -E_\theta\left[\frac{\partial^2}{\partial\theta^2} log(f(X,\theta))\right].$$

For the PIW distribution, the Fisher information (FI) that X contains about the parameters $\theta = (\alpha,\beta,\gamma)$ are obtained as follows: Using the pdf $g(x;\alpha,\beta,\gamma)$,

$$\ln g(x;\alpha,\beta,\gamma) = \ln\alpha\beta\gamma - (1+\beta)\ln(\alpha x) - \gamma(\alpha x)^{-\beta}.$$

We have the following partial derivatives:

$$\frac{\partial \ln g(x;\alpha,\beta,\gamma)}{\partial\alpha} = -\frac{\beta}{\alpha} + x^{-\beta}\beta\gamma\alpha^{-\beta-1},$$

$$\frac{\partial \ln g(x;\alpha,\beta,\gamma)}{\partial\beta} = \frac{1}{\beta} - \ln(\alpha x) + \gamma(\alpha x)^{-\beta}\ln(\alpha x),$$

$$\frac{\partial \ln g(x;\alpha,\beta,\gamma)}{\partial\gamma} = \frac{1}{\gamma} - (\alpha x)^{-\beta},$$

$$\frac{\partial^2 \ln g(x;\alpha,\beta,\gamma)}{\partial\alpha^2} = \frac{\beta}{\alpha^2} - \alpha^{-\beta-2}\beta(1+\beta)\gamma x^{-\beta},$$

$$\frac{\partial^2 \ln g(x;\alpha,\beta,\gamma)}{\partial\beta^2} = -\frac{1}{\beta^2} - \gamma(\alpha x)^{-\beta}\ln^2(\alpha x),$$

$$\frac{\partial^2 \ln g(x;\alpha,\beta,\gamma)}{\partial\gamma^2} = -\frac{1}{\gamma^2},$$

$$\frac{\partial^2 \ln g(x;\alpha,\beta,\gamma)}{\partial\alpha\partial\beta} = -\frac{1}{\alpha} + x^{-\beta}\beta\gamma\alpha^{-\beta-1}(-\ln(\alpha x) + \frac{1}{\beta}),$$

$$\frac{\partial^2 \ln g(x;\alpha,\beta,\gamma)}{\partial\alpha\partial\gamma} = x^{-\beta}\beta\alpha^{-\beta-1},$$

and

$$\frac{\partial^2 \ln g(x;\alpha,\beta,\gamma)}{\partial\beta\partial\gamma} = (\alpha x)^{-\beta}\ln(\alpha x).$$

Then,

$$\begin{aligned}
-E\left[\frac{\partial^2 \ln g(X;\alpha,\beta,\gamma)}{\partial\alpha^2}\right] &= -\int_0^\infty \left(\frac{\beta}{\alpha^2} - \alpha^{-\beta-2}\beta(1+\beta)\gamma x^{-\beta}\right) g(x;\alpha,\beta,\gamma)\,dx \\
&= \frac{\beta(1+\beta)}{\alpha^2}\int_0^\infty x\exp(-x)\,dx - \frac{\beta}{\alpha^2}, \\
&= \frac{\beta^2}{\alpha^2},
\end{aligned}$$

$$\begin{aligned}
-E\left[\frac{\partial^2 \ln g(X;\alpha,\beta,\gamma)}{\partial\beta^2}\right] &= -\int_0^\infty \left(-\frac{1}{\beta^2} - \gamma(\alpha x)^{-\beta}\ln^2(\alpha x)\right) g(x;\alpha,\beta,\gamma)\,dx \\
&= \int_0^\infty \left(\frac{1}{\beta^2} + \gamma(\alpha x)^{-\beta}\ln^2(\alpha x)\right)\frac{\alpha\beta\gamma e^{-\gamma(\alpha x)^{-\beta}}}{(\alpha x)^{\beta+1}}\,dx \\
&= \frac{1}{\beta^2} + \frac{1}{\beta^2}[\Gamma''(2) - 2\ln\gamma\cdot\Gamma'(2) + \ln^2\gamma] \\
&= \frac{1+\Gamma''(2) - 2\Gamma'(2)\ln\gamma + \ln^2\gamma}{\beta^2},
\end{aligned}$$

$$\begin{aligned}
-E\left[\frac{\partial^2 \ln g(X;\alpha,\beta,\gamma)}{\partial\gamma^2}\right] &= -\int_0^\infty \frac{\partial^2 \ln g(x;\alpha,\beta,\gamma)}{\partial\gamma^2} g(x;\alpha,\beta,\gamma)\,dx \\
&= -\int_0^\infty -\frac{1}{\gamma^2}\alpha\beta\gamma(\alpha x)^{-\beta-1}\exp[-\gamma(\alpha x)^{-\beta}]\,dx \\
&= \frac{1}{\gamma^2},
\end{aligned}$$

$$\begin{aligned}
-E\left[\frac{\partial^2 \ln g(X;\alpha,\beta,\gamma)}{\partial\alpha\partial\beta}\right] &= -\int_0^\infty \frac{\partial^2 \ln g(x;\alpha,\beta,\gamma)}{\partial\alpha\partial\beta} g(x;\alpha,\beta,\gamma)\,dx \\
&= \int_0^\infty [\frac{1}{\alpha} - x^{-\beta}\beta\gamma\alpha^{-\beta-1}(-\ln(\alpha x) + \frac{1}{\beta})]\frac{\alpha\beta\gamma e^{-\gamma(\alpha x)^{-\beta}}}{(\alpha x)^{\beta+1}} dx \\
&= \frac{\ln\gamma - \Gamma'(2)}{\alpha},
\end{aligned}$$

$$\begin{aligned}
-E\left[\frac{\partial^2 \ln g(X;\alpha,\beta,\gamma)}{\partial\alpha\partial\gamma}\right] &= -\int_0^\infty \frac{\partial^2 \ln g(x;\alpha,\beta,\gamma)}{\partial\alpha\partial\gamma} g(x;\alpha,\beta,\gamma)\,dx \\
&= -\int_0^\infty x^{-\beta}\beta\alpha^{-\beta-1}\alpha\beta\gamma(\alpha x)^{-\beta-1}\exp[-\gamma(\alpha x)^{-\beta}]\,dx \\
&= -\frac{\beta}{\alpha\gamma},
\end{aligned}$$

and

$$
\begin{aligned}
-E\left[\frac{\partial^2 \ln g(X;\alpha,\beta,\gamma)}{\partial\beta\partial\gamma}\right] &= -\int_0^\infty \frac{\partial^2 \ln g(x;\alpha,\beta,\gamma)}{\partial\beta\partial\gamma}\cdot g(x;\alpha,\beta,\gamma)\,dx \\
&= -\int_0^\infty (\alpha x)^{-\beta}\ln(\alpha x)\alpha\beta\gamma(\alpha x)^{-\beta-1}\exp[-\gamma(\alpha x)^{-\beta}]\,dx \\
&= \frac{1}{\beta\gamma}\int_0^\infty t[\ln t-\ln\gamma]\exp[-t]\,dt \\
&= \frac{\Gamma'(2)-\ln\gamma}{\beta\gamma}.
\end{aligned}
$$

Now, the Fisher Information Matrix (FIM) for PIW distribution is given by:

$$
I(\alpha,\beta,\gamma)=\begin{pmatrix}
-E\left[\frac{\partial^2 \log g(X;\alpha,\beta,\gamma)}{\partial\alpha^2}\right] & -E\left[\frac{\partial^2 \log g(X;\alpha,\beta,\gamma)}{\partial\alpha\partial\beta}\right] & -E\left[\frac{\partial^2 \log g(X;\alpha,\beta,\gamma)}{\partial\alpha\partial\gamma}\right] \\
-E\left[\frac{\partial^2 \log g(X;\alpha,\beta,\gamma)}{\partial\beta\partial\alpha}\right] & -E\left[\frac{\partial^2 \log g(X;\alpha,\beta,\gamma)}{\partial\beta^2}\right] & -E\left[\frac{\partial^2 \log g(X;\alpha,\beta,\gamma)}{\partial\beta\partial\gamma}\right] \\
-E\left[\frac{\partial^2 \log g(X;\alpha,\beta,\gamma)}{\partial\gamma\partial\alpha}\right] & -E\left[\frac{\partial^2 \log g(X;\alpha,\beta,\gamma)}{\partial\gamma\partial\beta}\right] & -E\left[\frac{\partial^2 \log g(X;\alpha,\beta,\gamma)}{\partial\gamma^2}\right]
\end{pmatrix},
$$

where the entries are given by

$$
I(1,1)=\frac{\beta^2}{\alpha^2},\quad I(1,2)=I(2,1)=\frac{\ln\gamma-\Gamma'(2)}{\alpha},
$$

$$
I(1,3)=I(3,1)=-\frac{\beta}{\alpha\gamma},
$$

$$
I(2,2)=\frac{1+\Gamma''(2)-2\Gamma'(2)\ln\gamma+\ln^2\gamma}{\beta^2},
$$

$$
I(2,3)=I(3,2)=\frac{\Gamma'(2)-\ln\gamma}{\beta\gamma},\quad \text{and}\quad I(3,3)=\frac{1}{\gamma^2}.
$$

5 Estimation of Parameters in the Proportional Inverse Weibull Distribution

In this section, we obtain estimates of the parameters for the PIW distribution. Method of moment (MOM) and maximum likelihood (ML) estimators are presented. Estimation of the parameters of PIW distribution for complete and right censored data are presented.

5.1 Method of Moment Estimators

Let $X_1, X_2, \cdots, X_n$ be an independent sample from the PIW distribution. The method of moments estimators are defined as follows:

$$
E(X^j)=\frac{\sum_{i=1}^n x_i^j}{n}\qquad j=1,2,\cdots.
$$

We have $E(X^j) = \gamma^{\frac{j}{\beta}} \alpha^{-j} \Gamma(\frac{\beta - j}{\beta})$, so we have the following equations:

$$\begin{aligned}
\gamma^{\frac{1}{\beta}} \alpha^{-1} \Gamma\Big(\frac{\beta-1}{\beta}\Big) &= \bar{X}, \\
\gamma^{\frac{2}{\beta}} \alpha^{-2} \Gamma\Big(\frac{\beta-2}{\beta}\Big) &= S^2 = \frac{\sum_{i=1}^n x_i^2}{n}, \\
\gamma^{\frac{3}{\beta}} \alpha^{-3} \Gamma\Big(\frac{\beta-3}{\beta}\Big) &= \triangle = \frac{\sum_{i=1}^n x_i^3}{n}.
\end{aligned}$$

From the division of the first two equations above, we get

$$\frac{\Gamma^2(\frac{\beta-1}{\beta})}{\Gamma(\frac{\beta-2}{\beta})} = \frac{\bar{X}^2}{S^2}.$$

Then we can apply Newton-Ralphson method to obtain the solution. Let $f(\beta) = \Gamma^2(\frac{\beta-1}{\beta})S^2 - \Gamma(\frac{\beta-2}{\beta})\bar{X}^2$, then do the iteration

$$\beta_{n+1} = \beta_n - \frac{f(\beta_n)}{f'(\beta_n)}$$

to find β, say $\hat{\beta}$. If α is known, then

$$\hat{\gamma} = \left(\frac{\alpha\bar{X}}{\Gamma(\frac{\hat{\beta}-1}{\hat{\beta}})}\right)^{\hat{\beta}}.$$

When γ is known,

$$\hat{\alpha} = \frac{\gamma^{\frac{1}{\hat{\beta}}} \Gamma(\frac{\hat{\beta}-1}{\hat{\beta}})}{\bar{X}}.$$

5.2 Maximum Likelihood Estimators

Let $X_1, X_2, \cdots, X_n$ be a random sample from a PIW distribution. Then the likelihood function is given by

$$\begin{aligned}
L &= g(x_1, \cdots, x_n; \alpha, \beta, \gamma) \\
&= (\alpha\beta\gamma)^n \Big(\prod_{i=1}^n \alpha x_i\Big)^{-\beta-1} \cdot \exp\Big[-\gamma \sum_{i=1}^n (\alpha x_i)^{-\beta}\Big].
\end{aligned}$$

The log-likelihood function is given by

$$\ln L = n \ln(\alpha\beta\gamma) - (1+\beta) \sum_{i=1}^n \ln(\alpha x_i) - \gamma \sum_{i=1}^n (\alpha x_i)^{-\beta}.$$

The normal equations are

$$
\begin{aligned}
\frac{\partial \ln L}{\partial \alpha} &= \frac{n}{\hat{\alpha}} - (1+\hat{\beta})\frac{n}{\hat{\alpha}} + \hat{\beta}\hat{\gamma}\sum_{i=1}^{n}(\hat{\alpha}x_i)^{-\hat{\beta}-1}x_i = 0,\\
\frac{\partial \ln L}{\partial \beta} &= \frac{n}{\hat{\beta}} - \sum_{i=1}^{n}\ln(\hat{\alpha}x_i) + \hat{\gamma}\sum_{i=1}^{n}[(\hat{\alpha}x_i)^{-\hat{\beta}}\ln(\hat{\alpha}x_i)] = 0,
\end{aligned}
$$

and

$$
\frac{\partial \ln L}{\partial \gamma} = \frac{n}{\hat{\gamma}} - \sum_{i=1}^{n}(\hat{\alpha}x_i)^{-\hat{\beta}} = 0,
$$

respectively. These equations reduces to

$$
\begin{aligned}
\frac{n}{\hat{\gamma}} &= \hat{\alpha}^{-\hat{\beta}}\sum_{i=1}^{n}x_i^{-\hat{\beta}}\\
\frac{n}{\hat{\beta}} &= \sum_{i=1}^{n}\ln(\hat{\alpha}x_i) - \hat{\gamma}\hat{\alpha}^{-\hat{\beta}}\sum_{i=1}^{n}[x_i^{-\hat{\beta}}\ln(\hat{\alpha}x_i)].
\end{aligned}
$$

From the normal equations, we know that if α or γ is known, we can use Newton's method to solve for β numerically. If γ and β are known, we solve for α to obtain

$$
\hat{\alpha} = \left(\frac{n}{\gamma\sum x_i^{-\beta}}\right)^{-\frac{1}{\beta}}.
$$

When α and β are known, we solve for γ to get

$$
\hat{\gamma} = \frac{n}{\alpha^{-\beta}\sum x_i^{-\beta}}.
$$

5.3 Numerical Examples

In this section, we provide several numerical examples to show and illustrate the flexibility of the generalized inverse Weibull distribution for date modeling. Specifically, we consider three data sets from Lawless (2003). The first set of data represents the number of million revolutions before failure of each of 22 ball bearing in a life testing experiment. The data are: 17.88, 28.92, 33.00, 41.52, 42.12, 45.60, 48.80, 51.84, 51.96, 54.12, 55.56, 67.80, 68.64, 68.64, 68.88, 84.12, 93.12, 98.64, 105.12, 127.92, 128.04, 173.40. The second data set represents the failure times (in minutes) for a sample of 15 electronic components in an accelerated life test and are given by: 1.4, 5.1, 6.3, 10.8, 12.1, 18.5, 19.7, 22.2, 23, 30.6, 37.3, 46.3, 53.9, 59.8, 66.2. The third set of data are the number of cycles of failure for 25 100-cm specimens of yarn, tested at a particular strain level. The data are: 15, 20, 38, 42, 61, 76, 86, 98, 121, 146, 149, 157, 175, 176, 180, 180, 198, 220, 224, 251, 264, 282, 321, 325, 653.

These data sets can be modeled by the generalized inverse Weibull distribution. The MLEs of the parameters α, β and γ are computed by maximizing the objective function with the trust-region algorithm via the NLPTR subroutine in SAS. The estimated values of the parameters α,

β and γ, log-likelihood statistic, Kolmogorov-Smirnov statistics and the corresponding gradient objective function (normal equations) under the generalized inverse Weibull distribution and other alternatives including the generalized Lindley distribution are presented in table 2. The generalized Lindley (GL) distribution (Zakerzadeh and Dolati (2009)) is given by

$$f_{GL}(x;\alpha,\beta,\gamma) = \frac{\beta^2(\beta x)^{\alpha-1}(\alpha+\gamma x)e^{-\beta x}}{(\gamma+\beta)\Gamma(\alpha+1)}, \quad \alpha,\beta,\gamma > 0.$$

The other models considered are the gamma, and lognormal distributions given by

$$f_G(x;\alpha,\beta) = (\Gamma(\alpha))^{-1}\beta^{\alpha}x^{\alpha-1}e^{-\beta x},$$

and

$$f_{LN}(x;\alpha,\beta) = \frac{1}{\sqrt{2\pi}\alpha x}e^{\frac{1}{2}(\frac{\log x-\beta}{\alpha})^2}.$$

After using NLPTR method, we have the following results for the generalized inverse Weibull parameter estimates for the first set of data: First, we set our initial guess as $\alpha = 10$, $\beta = 1$, and $\gamma = 10$. Then after 18 iterations, we have $\hat{\alpha} = 0.0983$, $\hat{\beta} = 1.8158$ and $\hat{\gamma} = 16.8395$. The value of the log likelihood function is -110.9885. For the second set of data, we have $\hat{\alpha} = 3.0001$, $\hat{\beta} = 0.8423$ and $\hat{\gamma} = 18.6706$. The value of the log likelihood function is -68.5351. For the third set of data, we have the following: $\hat{\alpha} = 0.1967$, $\hat{\beta} = 1.0111$ and $\hat{\gamma} = 16.6980$. The value of the log likelihood function is -158.5789. These values together with the values of the gradient object function for all three examples are presented in table 2.

5.4 Estimation in Right Censored Data from PIW Distribution

In this section, the maximum likelihood estimate (MLE) of the PIW distribution parameters α, β, and γ under type I censoring is presented. Suppose we have n independent positive random variables $X_1, X_2, \cdots, X_n$, where X_i has an associated indicator variable δ_i where $\delta_i = 1$ if X_i is an observed failure time and $\delta_i = 0$ if X_i is right censored, then the likelihood function is given by

$$L(x_1,x_2,\cdots,x_n;\theta) = \prod_{i=1}^{n} g^{\delta_i}(x_i;\alpha,\beta,\gamma)(1-G(x_i;\alpha,\beta,\gamma))^{1-\delta_i},$$

where $\theta = (\alpha,\beta,\gamma)$. The log-likelihood function is given by

$$\ln L(x_1,x_2,\cdots,x_n;\theta) = \sum_{i=1}^{n}\left[\delta_i \ln[g(x_i;\alpha,\beta,\gamma)] + (1-\delta_i)\ln[\bar{G}(x_i;\alpha,\beta,\gamma)]\right],$$

that is,

$$l(\theta) = \sum_{i=1}^{n}\left[\delta_i \ln(\alpha\beta\gamma) - \delta_i(\beta+1)\ln(\alpha x_i) - \delta_i\gamma(\alpha x_i)^{-\beta} - (1-\delta_i)\ln[1-\exp[-\gamma(\alpha x_i)^{-\beta}]]\right].$$

Table 2: **Estimates, Log-likelihood and Kolmogorov-Smirnov Statistic**

Data set	Model	α	β	γ	LL	K-S	$\frac{\partial L}{\partial \alpha}$	$\frac{\partial L}{\partial \beta}$	$\frac{\partial L}{\partial \gamma}$
I (n=22)	PIW	0.0983	1.8158	16.8395	-110.9885	0.091	-0.00000111	-0.000003387	0.000000214
	GL	0.0987	1.3708	0.09371	-108.6875	0.127	0.00000032	0.0000000579	0.000000327
	Gamma	0.0637	1.5646	-	-108.5738	0.116	0.00000055	-0.000000044	0.000000028
	Lognormal	3.1268	1.6732	-	-109.3792	0.176	0.000000032	0.0000000652	0.000000342
II (n=15)	PIW	3.0001	0.8423	18.6706	-68.5351	0.091	0.000000238	0.000000518	-0.00000004848
	GL	0.0654	1.2056	0.0851	-64.1001	0.102	0.000000044	0.000000479	-0.00000000433
	Gamma	0.0531	1.4434	-	-64.1880	0.100	0.0000000032	0.000000017	0.00000000392
	Lognormal	2.9295	1.0599	-	-65.6200	0.164	-0.0000005367	0.0000000359	-0.00000000043
III (n=25)	PIW	0.1967	1.0112	16.6980	-158.5789	0.121	-0.000006375	-0.000000948	0.0000000000
	GL	0.0134	1.5250	0.0185	-152.3741	0.141	0.0000000678	0.0000000896	0.0000000034
	Gamma	0.0113	1.7964	-	-152.4432	0.137	0.0000002340	-0.0000000044	0.0000000025
	Lognormal	4.9129	0.9031	-	-154.110	0.156	-0.000000523	-0.0000000361	0.00000000038

The MLEs $\hat{\theta} = (\hat{\alpha}, \hat{\beta}, \hat{\gamma})$ are obtained as the solution of the following system of equations:

$$\begin{aligned}
\frac{\partial l(\theta)}{\partial \alpha} &= \sum_{i=1}^{n} \left[-\frac{\delta_i \beta}{\alpha} + \delta_i \beta \gamma (\alpha x_i)^{-\beta-1} x_i + \frac{(1-\delta_i)\beta\gamma(\alpha x_i)^{-\beta-1} x_i h(x_i)}{1-h(x_i)} \right] = 0, \\
\frac{\partial l(\theta)}{\partial \beta} &= \sum_{i=1}^{n} \left[\frac{\delta_i}{\beta} - \delta_i \ln(\alpha x_i) + \delta_i \gamma (\alpha x_i)^{-\beta} \ln(\alpha x_i) + \frac{(1-\delta_i) h(x_i) \gamma (\alpha x_i)^{-\beta} \ln(\alpha x_i)}{1-h(x_i)} \right] = 0, \\
\frac{\partial l(\theta)}{\partial \gamma} &= \sum_{i=1}^{n} \left[\frac{\delta_i}{\gamma} - \delta_i (\alpha x_i)^{-\beta} - \frac{(1-\delta_i) h(x_i) (\alpha x_i)^{-\beta}}{1-h(x_i)} \right] = 0,
\end{aligned}$$

where $h(x) = h(x; \alpha, \beta, \gamma) = \exp[-\gamma(\alpha x)^{-\beta}]$. The system does not admit any explicit solution, therefore the ML estimates $\hat{\theta} = (\hat{\alpha}, \hat{\beta}, \hat{\gamma})$ can be obtained only by means of numerical procedures.

Under the usual regularity conditions, the well-known asymptotic properties of the maximum likelihood method ensure that $\sqrt{n}\left(\hat{\theta} - \theta\right) \rightarrow_d N(\mathbf{0}, \Sigma_\theta)$, where $\Sigma_\theta = [\mathbf{I}(\theta)]^{-1}$ is the asymptotic variance-covariance matrix and $\mathbf{I}(\theta)$ is the Fisher Information Matrix, whose entries were calculated earlier.

6 Generalization via Kumaraswamy Distribution

In this section, we present results on the generalized inverse Weibull Distribution via the Kumaraswamy distribution. In particular, we derive the probability density function (pdf), cumulative distribution function (cdf), moments, and some additional properties.

6.1 Cumulative Distribution and Probability Density Functions

The cdf and pdf of the Kumaraswamy proportional inverse Weibull (Kum-PIW) distribution are given by

$$\begin{aligned}
G_k(x; \alpha, \beta, \gamma, \lambda, \varphi) &= G_k(x) \\
&= 1 - [1 - G^{\lambda}(x)]^{\varphi} \\
&= 1 - \{1 - \exp[-\gamma\lambda(\alpha x)^{-\beta}]\}^{\varphi},
\end{aligned}$$

and

$$\begin{aligned}
g_k(x; \alpha, \beta, \gamma, \lambda, \varphi) &= g_k(x) \\
&= \alpha\beta\gamma\lambda\varphi(\alpha x)^{-\beta-1} \exp[-\gamma\lambda(\alpha x)^{-\beta}]\{1 - \exp[-\gamma\lambda(\alpha x)^{-\beta}]\}^{\varphi-1},
\end{aligned}$$

for $\alpha > 0$, $\beta > 0$, $\gamma > 0$, $\lambda > 0$, and $\varphi > 0$ respectively.

6.2 Mode of Kum-PIW Distribution

To obtain the mode of the Kum-PIW distribution, we solve the equation $\frac{\partial \ln g_k(x; \alpha, \beta, \gamma, \lambda, \varphi)}{\partial x} = 0$ for x. Note that

$$\ln g_k(x) = \ln(\alpha\beta\gamma\lambda\varphi) - (1+\beta)\ln(\alpha x) - \gamma\lambda(\alpha x)^{-\beta} + (\varphi - 1)\ln\{1 - \exp[-\gamma\lambda(\alpha x)^{-\beta}]\}.$$

The derivative of $\ln g_k(x;\alpha,\beta,\gamma,\lambda,\varphi)$ with respect to x is given by

$$\frac{\partial \ln g_k(x)}{\partial x} = -\frac{1+\beta}{x}+\alpha\beta\gamma\lambda(\alpha x)^{-\beta-1}+\frac{(\varphi-1)\{-\exp[-\gamma\lambda(\alpha x)^{-\beta}]\alpha\beta\gamma\lambda(\alpha x)^{-\beta-1}\}}{1-\exp[-\gamma\lambda(\alpha x)^{-\beta}]}.$$

Now $\frac{\partial \ln g_k(x;\alpha,\beta,\gamma,\lambda,\varphi)}{\partial x}=0$ implies

$$-(1+\beta)\{1-\exp[-\gamma\lambda(\alpha x)^{-\beta}]\}+\beta\gamma\lambda(\alpha x)^{-\beta}\{1-\varphi\exp[-\gamma\lambda(\alpha x)^{-\beta}]\}=0.$$

When $\varphi=1$, the above equation reduces to:

$$\{1-\exp[-\gamma\lambda(\alpha x)^{-\beta}]\}[\beta\gamma\lambda(\alpha x)^{-\beta}-1-\beta]=0,$$

and we obtain $x=\left(\frac{\alpha^{-\beta}\beta\gamma\lambda}{1+\beta}\right)^{\frac{1}{\beta}}$, which the same result as the mode of PIW distribution when $\lambda=1$. When $\varphi\neq 1$, we can use numerical method to solve for the mode.

6.3 Hazard Function

The hazard function of the Kum-PIW distribution is given by

$$\begin{aligned}\lambda_{G_k}(x;\alpha,\beta,\gamma,\lambda,\varphi) &= \frac{g_k(x;\alpha,\beta,\gamma,\lambda,\varphi)}{\bar{G}_k(x;\alpha,\beta,\gamma,\lambda,\varphi)}\\ &= \frac{\alpha\beta\gamma\lambda\varphi(\alpha x)^{-\beta-1}\exp[-\gamma\lambda(\alpha x)^{-\beta}]}{1-\exp[-\gamma\lambda(\alpha x)^{-\beta}]},\end{aligned}$$

for $\alpha>0$, $\beta>0$, $\lambda>0$, $\gamma>0$, and $\varphi>0$. We can apply Glaser's Theorem to see that the hazard function of Kum-PIWD has an upside down bathtub shape (UBT).

6.4 Reverse Hazard Function

The reverse hazard function of the Kum-PIW distribution is given by

$$\begin{aligned}\tau_{G_k}(x;\alpha,\beta,\gamma,\lambda,\varphi) &= \frac{g_k(x;\alpha,\beta,\gamma,\lambda,\varphi)}{G_k(x;\alpha,\beta,\gamma,\lambda,\varphi)}\\ &= \frac{\alpha\beta\gamma\lambda\varphi(\alpha x)^{-\beta-1}e^{[-\gamma\lambda(\alpha x)^{-\beta}]}\{1-e^{[-\gamma\lambda(\alpha x)^{-\beta}]}\}^{\varphi-1}}{1-\{1-e^{[-\gamma\lambda(\alpha x)^{-\beta}]}\}^{\varphi}},\end{aligned}$$

for $\alpha>0$, $\beta>0$, $\lambda>0$, $\gamma>0$, and $\varphi>0$.

6.5 Moments

The c^{th} non-central moment of the Kum-PIW distribution is given by

$$\begin{aligned}E(X^c) &= \int_0^\infty x^c g_k(x;\alpha,\beta,\gamma,\lambda,\varphi)\,dx\\ &= \int_0^\infty x^c\alpha\beta\gamma\lambda\varphi(\alpha x)^{-\beta-1}\exp[-\gamma\lambda(\alpha x)^{-\beta}]\{1-\exp[-\gamma\lambda(\alpha x)^{-\beta}]\}^{\varphi-1}\,dx.\end{aligned}$$

Note that when $\varphi = 1$, this turns out to be the moments of the PIW distribution with parameters $\alpha' = \alpha$, $\beta' = \beta$ and $\gamma' = \gamma\lambda$, that is

$$E(X^c) = (\gamma\lambda)^{\frac{c}{\beta}} \alpha^{-c} \Gamma\left(\frac{\beta - c}{\beta}\right), \quad \text{for } \beta > c.$$

When $\varphi > 1$ and an integer, note that

$$\{1 - \exp[-\gamma\lambda(\alpha x)^{-\beta}]\}^{\varphi-1} = \sum_{k=0}^{\varphi-1} \binom{\varphi-1}{k} (-1)^k \exp[-\gamma\lambda k(\alpha x)^{-\beta}],$$

and we have

$$\begin{aligned} E(X^c) &= \int_0^\infty x^c \alpha\beta\gamma\lambda\varphi(\alpha x)^{-\beta-1} \exp[-\gamma\lambda(\alpha x)^{-\beta}] \sum_{k=0}^{\varphi-1} \binom{\varphi-1}{k} (-1)^k \exp[-\gamma\lambda k(\alpha x)^{-\beta}]\, dx \\ &= \sum_{k=0}^{\varphi-1} \int_0^\infty x^c \alpha\beta\gamma\lambda\varphi(\alpha x)^{-\beta-1} \exp[-\gamma\lambda(\alpha x)^{-\beta}] \binom{\varphi-1}{k} (-1)^k \exp[-\gamma\lambda k(\alpha x)^{-\beta}]\, dx \\ &= \sum_{k=0}^{\varphi-1} \{\alpha\beta\gamma\lambda\varphi(-1)^k \binom{\varphi-1}{k} \int_0^\infty x^c (\alpha x)^{-\beta-1} \exp[-\gamma\lambda(1+k)(\alpha x)^{-\beta}]\, dx\}. \end{aligned}$$

Let $\gamma\lambda(1+k)(\alpha x)^{-\beta} = t$, then we have

$$\begin{aligned} E(X^c) &= \sum_{k=0}^{\varphi-1} \left(\frac{\varphi(-1)^k \binom{\varphi-1}{k}}{\alpha^c} (1+k)^{\frac{c}{\beta}-1} (\gamma\lambda)^{\frac{c}{\beta}} \int_0^\infty t^{\frac{\beta-c}{\beta}-1} \exp[-t]\, dt \right) \\ &= \frac{(\gamma\lambda)^{\frac{c}{\beta}} \varphi}{\alpha^c} \Gamma\left(\frac{\beta-c}{\beta}\right) \sum_{k=0}^{\varphi-1} \left((-1)^k \binom{\varphi-1}{k} (1+k)^{\frac{c}{\beta}-1} \right). \end{aligned}$$

6.6 Shannon Entropy

Shannon entropy for Kum-PIW distribution is given by

$$\begin{aligned} H(g_k) &= E[-\ln g_k(X)] \\ &= -\int_0^\infty \ln g_k(x) \cdot g_k(x)\, dx \\ &= -(A + B + C + D), \end{aligned}$$

where

$$\begin{aligned} A &= \int_0^\infty \ln(\alpha\beta\gamma\lambda\varphi) \cdot g_k(x)\, dx \\ &= \ln(\alpha\beta\gamma\lambda\varphi), \end{aligned}$$

$$B = \int_0^\infty -(1+\beta)\ln(\alpha x)\alpha\beta\gamma\lambda\varphi(\alpha x)^{-\beta-1} \exp[-\gamma\lambda(\alpha x)^{-\beta}] \cdot \{1 - \exp[-\gamma\lambda(\alpha x)^{-\beta}]\}^{\varphi-1}\, dx,$$

$C=\int_0^\infty -\gamma\lambda(\alpha x)^{-\beta} g_k(x)dx$, and

$$D=\int_0^\infty (\varphi-1)\ln\{1-\exp[-\gamma\lambda(\alpha x)^{-\beta}]\}\cdot g_k(x)\,dx.$$

We simplify the quantities A, B, C, and D. Note that

$$\begin{aligned} B &= \sum_{k=0}^{\varphi-1}\left[-(1+\beta)\alpha\beta\gamma\lambda\varphi\binom{\varphi-1}{k}(-1)^k\right] \\ &\times \int_0^\infty \ln(\alpha x)\cdot(\alpha x)^{-\beta-1}\exp[-\gamma\lambda(k+1)(\alpha x)^{-\beta}]\,dx. \end{aligned}$$

Let $\gamma\lambda(k+1)(\alpha x)^{-\beta}=t$, then the quantity B can be rewritten as

$$\begin{aligned} B &= \sum_{k=0}^{\varphi-1}\frac{(1+\beta)\varphi\binom{\varphi-1}{k}(-1)^k}{\beta(1+k)}\int_0^\infty[\ln t-\ln(\gamma\lambda(1+k))]\exp[-t]\,dt \\ &= \sum_{k=0}^{\varphi-1}\frac{(1+\beta)\varphi\binom{\varphi-1}{k}(-1)^k}{\beta(1+k)}[\Gamma'(1)-\ln(\gamma\lambda(1+k))]. \end{aligned}$$

Also, for the quantity C, we have

$$\begin{aligned} C &= \int_0^\infty -\gamma\lambda(\alpha x)^{-\beta}\cdot g_k(x)\,dx \\ &= -\alpha\beta\gamma^2\lambda^2\varphi\int_0^\infty(\alpha x)^{-2\beta-1}\exp[-\gamma\lambda(\alpha x)^{-\beta}]\{1-\exp[-\gamma\lambda(\alpha x)^{-\beta}]\}^{\varphi-1}\,dx \\ &= -\alpha\beta\gamma^2\lambda^2\varphi\sum_{k=0}^{\varphi-1}\left[\binom{\varphi-1}{k}(-1)^k\cdot\int_0^\infty(\alpha x)^{-2\beta-1}\exp[-\gamma\lambda(1+k)(\alpha x)^{-\beta}]\,dx\right]. \end{aligned}$$

Now, let $\gamma\lambda(k+1)(\alpha x)^{-\beta}=t$, then C reduces to

$$\begin{aligned} C &= -\varphi\sum_{k=0}^{\varphi-1}\left[\frac{\binom{\varphi-1}{k}(-1)^k}{(1+k)^2}\int_0^\infty t\exp[-t]\,dt\right] \\ &= -\varphi\sum_{k=0}^{\varphi-1}\left[\frac{\binom{\varphi-1}{k}(-1)^k}{(1+k)^2}\right]. \end{aligned}$$

Note that

$$\begin{aligned} D &= \int_0^\infty(\varphi-1)\ln\{1-\exp[-\gamma\lambda(\alpha x)^{-\beta}]\}\cdot g_k(x)\,dx \\ &= (\varphi-1)\alpha\beta\gamma\lambda\varphi\sum_{k=0}^{\varphi-1}\binom{\varphi-1}{k}(-1)^k\cdot \\ &\int_0^\infty(\alpha x)^{-\beta-1}\ln\{1-\exp[-\gamma\lambda(\alpha x)^{-\beta}]\}\cdot\exp[-\gamma\lambda(1+k)(\alpha x)^{-\beta}]\,dx. \end{aligned}$$

To simplify D, we let $\gamma\lambda(k+1)(\alpha x)^{-\beta} = t$, then

$$D = (\varphi-1)\varphi\sum_{k=0}^{\varphi-1}\left[\frac{\binom{\varphi-1}{k}(-1)^k}{1+k}\cdot\int_0^\infty \ln\left(1-\exp\left[-\frac{t}{1+k}\right]\right)\exp[-t]\,dt\right].$$

We know that Taylor expansion for $\ln(1-x)$ is:

$$\ln(1-x) = -\sum_{n=1}^{\infty}\frac{x^n}{n}, \qquad \text{for } |x|<1,$$

so that

$$\begin{aligned}\int_0^\infty \ln\left(1-\exp\left[-\frac{t}{1+k}\right]\right)\exp[-t]\,dt &= \int_0^\infty -\sum_{n=1}^{\infty}\frac{\exp[-\frac{nt}{1+k}]}{n}\exp[-t]\,dt\\ &= -\sum_{n=1}^{\infty}\int_0^\infty \frac{\exp[-\frac{(1+k+n)t}{1+k}]}{n}\,dt\\ &= -\sum_{n=1}^{\infty}\frac{1+k}{n(1+k+n)}\\ &= -\sum_{n=1}^{k+1}\frac{1}{n},\end{aligned}$$

and

$$\begin{aligned}D &= (\varphi-1)\varphi\sum_{k=0}^{\varphi-1}\left[\frac{\binom{\varphi-1}{k}(-1)^k}{1+k}\cdot(-\sum_{n=1}^{k+1}\frac{1}{n})\right]\\ &= -(\varphi-1)\varphi\sum_{k=0}^{\varphi-1}\left[\frac{\binom{\varphi-1}{k}(-1)^k}{1+k}\cdot\sum_{n=1}^{k+1}\frac{1}{n}\right].\end{aligned}$$

Finally, we obtain Shannon entropy:

$$H(g_k(x)) = -[A+B+C+D],$$

that is,

$$H(g_k(x)) = \sum_{k=0}^{\varphi-1}\left(\frac{\varphi\binom{\varphi-1}{k}(-1)^k}{1+k}\left[\frac{1}{1+k}+(\varphi-1)\sum_{n=1}^{k+1}\frac{1}{n}-\frac{1+\beta}{\beta}[\Gamma'(1)-\ln[\gamma\lambda(1+k)]]\right]\right)$$
$$-\ln(\alpha\beta\gamma\lambda\varphi).$$

6.7 Estimation in Right Censored Data from Kum-PIW Distribution

In this section, the maximum likelihood estimation (MLE) of the Kum-PIW distribution parameters α, β, γ, λ and φ under type I censoring is presented. Suppose we have n independent positive random variables $X_1, X_2, \cdots, X_n$ where X_i has an associated indicator variable δ_i where

$\delta_i = 1$ if X_i is an observed failure time and $\delta_i = 0$ if X_i is right censored, then the likelihood function is given by

$$L = \prod_{i=1}^{n} g_k^{\delta_i}(x_i;\alpha,\beta,\gamma,\lambda,\varphi)\,[1-G_k(x_i;\alpha,\beta,\gamma,\lambda,\varphi)]^{1-\delta_i}.$$

The log-likelihood function is,

$$\ln L = \sum_{i=1}^{n}\{\delta_i \ln[g_k(x_i;\alpha,\beta,\gamma,\lambda,\varphi)] + (1-\delta_i)\ln[1-G_k(x_i;\alpha,\beta,\gamma,\lambda,\varphi)]\},$$

that is

$$l(\theta) = \ln L = \sum_{i=1}^{n}\left[\delta_i \ln(\alpha\beta\gamma\lambda\varphi) - \delta_i(\beta+1)\ln(\alpha x_i) + \delta_i \ln h(x_i) + (\varphi-\delta_i)\ln[1-h(x_i)]\right],$$

where $h(x) = \exp[-\gamma\lambda(\alpha x)^{-\beta}]$.

The derivative of $h(x_i)$ with respect to the parameters are:

$$\begin{aligned}
\frac{\partial h(x_i)}{\partial \alpha} &= h(x_i)[\beta\gamma\lambda(\alpha x_i)^{-\beta-1}x_i],\\
\frac{\partial h(x_i)}{\partial \beta} &= h(x_i)[\gamma\lambda(\alpha x_i)^{-\beta}\ln(\alpha x_i)],\\
\frac{\partial h(x_i)}{\partial \gamma} &= h(x_i)[-\lambda(\alpha x_i)^{-\beta}],\\
\frac{\partial h(x_i)}{\partial \lambda} &= h(x_i)[-\gamma(\alpha x_i)^{-\beta}].
\end{aligned}$$

The normal equations are given by:

$$\begin{aligned}
\frac{\partial l(\theta)}{\partial \alpha} &= \sum_{i=1}^{n}\{-\frac{\delta_i\beta}{\alpha} + B(x_i,\delta_i)\beta\gamma\lambda(\alpha x_i)^{-\beta-1}x_i\} = 0,\\
\frac{\partial l(\theta)}{\partial \beta} &= \sum_{i=1}^{n}\{\frac{\delta_i}{\beta} - \delta_i\ln(\alpha x_i) + B(x_i,\delta_i)\gamma\lambda(\alpha x_i)^{-\beta}\ln(\alpha x_i)\} = 0,\\
\frac{\partial l(\theta)}{\partial \gamma} &= \sum_{i=1}^{n}\{\frac{\delta_i}{\gamma} + B(x_i,\delta_i)[-\lambda(\alpha x_i)^{-\beta}]\} = 0,\\
\frac{\partial l(\theta)}{\partial \lambda} &= \sum_{i=1}^{n}\{\frac{\delta_i}{\lambda} + B(x_i,\delta_i)[-\gamma(\alpha x_i)^{-\beta}]\} = 0,\\
\frac{\partial l(\theta)}{\partial \varphi} &= \sum_{i=1}^{n}\{\frac{\delta_i}{\varphi} + \ln[1-h(x_i)]\} = 0,
\end{aligned}$$

where $B(x_i,\delta_i) = \frac{\delta_i - \varphi h(x_i)}{1-h(x_i)}$. These equations do not have a closed form solution. We can use numerical methods to solve this problem.

7 Concluding Remarks

Results on the generalized inverse Weibull and Kumaraswamy generalized inverse Weibull distributions are presented. This class of distributions contains a fairly large number of distributions with potential applications to a wide area of probability and statistics. Properties of the generalized inverse Weibull and Kumaraswamy generalized inverse Weibull distributions including the pdfs, cdfs, moments, hazard functions, reverse hazard functions, coefficients of variation, skewness and kurtosis, Fisher information, Shannon entropy and β-entropy are presented. Estimation of the parameters of the models for complete and right censored data are also presented.

Acknowledgment

The authors wish to express their gratitude to the referees and editor for their valuable and insightful comments.

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