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Estimation of Parameters of Exponentiated Pareto Distribution for Progressive Type-II Censored Data with Binomial Random Removals Scheme

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In this paper, we propose maximum likelihood estimators and Bayes estimators of parameters of exponentiated Pareto distribution under general Entropy loss function and squared error loss function for Progressive type-II censored data with binomial removals. The maximum likelihood estimators and corresponding Bayes estimators are compared in terms of their risks based on simulated samples from exponentiated Pareto distribution

keywords: Maximum likelihood estimators, exponentiated Pareto distribution under general Entropy loss function, Progressive type-II censored data with binomial removals .

1 Introduction

In life testing experiments, situations do arise when units are lost or removed from the experiments while they are still alive; i.e, we get censored data from the experiment. The loss of units may occur due to time constraints giving type-I censored data. In such censoring scheme, experiment is terminated at specified time. Sometimes, the experiment is terminated after a prefixed number of observations due to cost constraints

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and we get type-II censored data. Besides these two controlled causes, units may drop out of the experiment randomly due to some uncontrolled causes. For example, consider that a doctor perform an experiment with n cancer patients but after the death of first patient, some patient leave the experiment and go for treatment to other doctor/ hospital. Similarly, after the second death a few more leave and so on. Finally the doctor stops taking observation as soon as the predetermined number of deaths (say m) are recorded. It may be assumed here that each stage the participating patients may independently decide to leave the experiment and the probability (p) of leaving the experiment is same for all the patients. Thus the number of patients who leave the experiment at a specified stage will follow binomial distribution with probability of success (p). The experiment is similar to a life test experiment which starts with n units. At the first failure X_1, r_1 (random) units are removed from the remaining $(n-1)$ surviving units. At second failure X_2, r_2 unit from remaining $n-2-r_1$ units are removed, and so on, till m^{th} failure is observed; i.e., at m^{th} failure all the remaining $r_m = n - m - r_1 - r_2 \cdots r_{m-1}$ units are removed. It may be re-emphasized that, here, m is pre-fixed constant and $r_i^{'}s$ are random. Such a censoring mechanism is termed as progressive type-II censoring with random removal scheme. As stated above, if we assume that probability of removal of a unit, at every stage, is p for each unit then r_i can be considered to follow a Binomial distribution; i.e, $r_i \approx B(n-m-\sum_{l=0}^{i-1} r_l, p)$ for $i = 1, 2, 3, \cdots m-1$ with $r_0 = 0$. For further details, reader are referred to Balakrishnan (2007). In last few years, the estimation of parameters of different life time distribution based on progressive censored samples have been studied by several authors such as Childs and Balakrishnan (2000), Balakrishnan and Kannan (2001), Mousa and Jaheen (2002), Chan and Balakrishnan (2002), Sarhan and Abuamooh (2008), Ashour and Afify (2007) and Ashour and Afify (2008). The progressive type-II censoring with binomial removal has been considered by Tse, Yang and Yuen (2000) for Weibull distribution, Wu and Chang (2002) for Exponential distribution. Under the progressive type-II censoring with random removals, Wu and Chang (2003) and Yuen and Tse (1996) developed the estimation problem for the Pareto distribution and Weibull distribution respectively, when the number of units removed at each failure time has a discrete uniform distribution, the expected time of this censoring plan is discussed and compared numerically.

Let us assume that the life time of the units follow the exponentiated Pareto distribution (EPD) with cumulative distribution function

$$F(x, \alpha, \theta) = \left[1 - (1+x)^{-\alpha}\right]^{\theta} \quad ; \quad x > 0, \quad \alpha > 0, \quad \theta > 0 \tag{1}$$

This distribution was introduced by Gupta, Gupta and Gupta (1998). The probability density function (pdf) of X takes the following form with two shape parameters α and θ:

$$f(x, \alpha, \theta) = \alpha\theta \left[1 - (1+x)^{-\alpha}\right]^{\theta-1} (1+x)^{-(\alpha+1)}; \quad x > 0, \quad \alpha > 0, \quad \theta > 0 \tag{2}$$

The corresponding survival function is

$$S(x) = 1 - F(x, \alpha, \theta) = 1 - \left[1 - (1+x)^{-\alpha}\right]^{\theta} \quad ; \quad x > 0, \quad \alpha > 0, \quad \theta > 0 \tag{3}$$

It may be noted that for $\theta = 1$, (2) reduces to

$$f_1(x, \alpha, \theta) = \alpha(1 + x)^{-(\alpha+1)} \quad ; \quad x > 0, \quad \alpha > 0 \tag{4}$$

which is standard Pareto distribution of second kind Kotz and Balakrishnan (1994). The estimators of the parameters of exponentiated Pareto distribution have been obtained by Shawky, Abu-Zinadah and Hanna (2009) under different estimation procedures for complete sample case. The estimation of parameters has also been attempted by Afify (2010) under type-I and type-II censoring scheme. However, no attempt has been made to develop estimators for the parameters of EPD under progressive type-II censoring with binomial removals. Therefore, we propose to develop such an estimation procedure. Rest of the paper is organized as follows:

Section 2 provides the likelihood function. In section 3, maximum likelihood estimator (MLE) and Bayes estimators have been obtained. An algorithm for simulating the progressive type-II sample with binomial removal is presented in section 4. For illustration purpose estimates have been obtained for a simulated data in section 5. The comparison of MLE's and corresponding Bayes estimators are given in section 6. Comparisons are based on simulation studies of risk (average loss over sample space) of the estimators. Finally conclusions are provided in the last section.

2 Likelihood Function

Let $(X_1, R_1), (X_2, R_2), (X_3, R_3), \cdots, (X_m, R_m)$, denote a random progressive type II censored sample from (2), where $X_1 < X_2 < X_3, \cdots < X_m$. For given number of removals, say $R_1 = r_1, R_2 = r_2, R_3 = r_3, \cdots, R_m = r_m$, the conditional likelihood function can be written as (see Cohen (1963)):

$$L(\alpha, \theta; x|R = r) = c^* \prod_{i=1}^{m} f(x_i) [S(x_i)]^{r_i} \tag{5}$$

where $c^* = n(n-r_1-1)(n-r_1-r_2-2)(n-r_1-r_2-r_3-3)\cdots(n-r_1-r_2-r_3, \cdots, r_m-m+1)$, and $0 \leq r_i \leq (n - m - r_1 - r_2 - r_3 \cdots r_{i-1})$, for $i = 1, 2, 3 \ldots, m-1$ and $r_0 = 0$. Substituting (2) and (3) into (5), we get

$$L(\alpha, \theta; x|R = r) = c^* \prod_{i=1}^{m} \alpha\theta \left[1 - (1 + x_i)^{-\alpha}\right]^{\theta-1} \left\{1 - \left[1 - (1 + x_i)^{-\alpha}\right]^{\theta}\right\}^{r_i} (1 + x_i)^{-(\alpha+1)} \tag{6}$$

We assume that units removed from the test at i^{th} failure, $i = 1, 2, \cdots, m-1$, are independent of each other. However, probability of removal of unit remains same (say, p) for all units at all failures; i.e., R_i (the number of units removed at the i^{th} failure $i = 1, 2, 3, \cdots, m-1$) follows binomial distribution with parameters $n - m - \sum_{l=0}^{i-1} r_l$ and p. Therefore,

$$P(R_1 = r_1) = \binom{n-m}{r_1} p^{r_1} (1-p)^{n-m-r_1}, \tag{7}$$

and for $i = 2, 3, \cdots, m-1$,

$$P\left(R_i = r_i | R_{i-1} = r_{i-1}, \cdots R_1 = r_1\right) = \binom{n - m - \sum_{l=0}^{i-1} r_l}{r_i} p^{r_i} (1-p)^{n-m-\sum_{l=0}^{i-1} r_l} \quad (8)$$

We further assume that R_i is independent of X_i for all i. Hence the likelihood function takes the following form

$$L\left(\alpha, \theta, p; x\right) = L\left(\alpha, \theta; x | R = r\right) P\left(R = r\right), \quad (9)$$

where

$$\begin{aligned} P\left(R = r\right) = P\left(R_1 = r_1\right) P\left(R_2 = r_2 | R_1 = r_1\right) P\left(R_3 = r_3 | R_2 = r_2, R_1 = r_1\right) \\ \cdots P\left(R_{m-1} = r_{m-1} | R_{m-2} = r_{m-2}, \cdots R_1 = r_1\right). \end{aligned} \quad (10)$$

Substituting (7) and (8) into (10), we get

$$P\left(R = r\right) = \frac{(n-m)! p^{\sum_{i=1}^{m-1} r_i} (1-p)^{(m-1)(n-m) - \sum_{i=1}^{m-1} (m-i) r_i}}{\left(n - m - \sum_{l=1}^{i-1} r_i\right)! \prod_{i=1}^{m-1} r_i!} \quad (11)$$

Using (6),(9) and (11), we can write the likelihood function in the following form :

$$L\left(\alpha, \theta, p; x\right) = A L_1\left(\alpha, \theta\right) L_2\left(p\right), \quad (12)$$

where
$A = \frac{c^* (n-m)!}{\left(n-m-\sum_{l=1}^{i-1} r_i\right)! \prod_{i=1}^{m-1} r_i!}$,

$$L_1(\alpha; \theta) = \prod_{i=1}^{m} \alpha\theta \left[1 - (1 + x_i)^{-\alpha}\right]^{\theta - 1} \left\{1 - \left[1 - (1 + x_i)^{-\alpha}\right]^{\theta}\right\}^{r_i} (1 + x_i)^{-(\alpha+1)} \quad (13)$$

and

$$L_2\left(p\right) = p^{\sum_{i=1}^{m-1} r_i} (1-p)^{(m-1)(n-m) - \sum_{i=1}^{m-1} (m-i) r_i}. \quad (14)$$

3 Classical and Bayesian Estimation of Parameters

3.1 Maximum Likelihood Estimation

In this section, we have obtained the MLE of the parameters θ, α and p based on progressive type-II censored data with binomial removals. We observe from (12), (13) and (14) that likelihood function is multiplication of three terms, namely, A, L_1 and L_2. Out of these, A does not dependent on the parameters α, θ and p; thus, it behaves as constant for maximum likelihood estimation. L_1 does not involved p and can be treated as function of α and θ only, where as L_2 involves p only. Therefore, the MLE's of α and θ can be derived by maximizing L_1 with respect to α and θ. Similarly, the MLE of p can be obtained by maximizing L_2.

Taking log of both sides of (13), we have

$$\begin{aligned}\ln L_1(\alpha;\theta) = {} & m\ln(\theta) + m\ln(\alpha) \\ & + (\theta-1)\sum_{i=1}^{m}\ln\left[1-(1+x_i)^{-\alpha}\right] \\ & + \sum_{i=1}^{m} r_i \ln\left[1-\left[1-(1+x_i)^{-\alpha}\right]^{\theta}\right] \\ & - (\alpha+1)\sum_{i=1}^{m}\ln(1+x_i)\end{aligned} \tag{15}$$

Thus, the normal equations can be obtained by differentiating (15) with respect to α and θ and equating these to zero; i.e., MLE's $\hat{\alpha}$ and $\hat{\theta}$ of α and θ respectively, can be obtained by simultaneously solving the following normal equations:

$$\begin{aligned}&\frac{m}{\alpha} + (\theta-1)\sum_{i=1}^{m}\frac{(1+x_i)^{-\alpha}\ln(1+x_i)}{1-(1+x_i)^{-\alpha}} - \sum_{i=1}^{m}\ln(1+x_i) - \\ &\theta\sum_{i=1}^{m}\frac{r_i\left[1-(1+x_i)^{-\alpha}\right]^{\theta-1}(1+x_i)^{-\alpha}\ln(1+x_i)}{1-\left[1-(1+x_i)^{-\alpha}\right]^{\theta}} = 0\end{aligned} \tag{16}$$

and

$$\begin{aligned}&\frac{m}{\theta} + \sum_{i=1}^{m}\ln\left[1-(1+x_i)^{-\alpha}\right] \\ &\sum_{i=1}^{m}\frac{r_i\left[1-(1+x_i)^{-\alpha}\right]^{\theta}\ln\left[1-(1+x_i)^{-\alpha}\right]}{1-\left[1-(1+x_i)^{-\alpha}\right]^{\theta}} = 0\end{aligned} \tag{17}$$

It may be noted that (16) and (17) can not be solved simultaneously to provide a nice closed form for the estimators. Therefore, we propose to use fixed point iteration method for solving these equations numerically. For details about the proposed method readers may refer Jain, Iyengar and Jain (1984). When procedure for obtaining the iteration function and the choice of initial guesses, based on maximum absolute row sum norms, have been discussed.

The log of $L_2(p)$ takes the following form

$$\ln L_2(p) = \ln p\sum_{i=1}^{m-1} r_i + \ln(1-p)\left[(m-1)(n-m) - \sum_{i=1}^{m-1} r_i(m-i)\right]. \tag{18}$$

The first order derivative of $\ln L_2(p)$ with respect to p is

$$\frac{\partial \ln L_2(p)}{\partial p} = \frac{\sum_{i=1}^{m-1} r_i}{p} - \frac{(m-1)(n-m) - \sum_{i=1}^{m-1} r_i(m-i)}{1-p}. \tag{19}$$

Setting $\frac{\partial \ln L_2(p)}{\partial p} = 0$, we get the normal equation for p. Solving this equation for p, we get the MLE of p as

$$\hat{p}_M = \frac{\sum_{i=1}^{m-1} r_i}{(m-1)(n-m) - \sum_{i=1}^{m-1} r_i (m-i-1)}. \tag{20}$$

3.2 Bayes procedure

In this section, we have obtained the Bayes estimators of the parameters α, θ and p based on progressively type-II censored data with Binomial removals. In order to obtain the Bayes estimator, we must assume that the parameters α, θ and p are random variables. We further assume that these are independently distributed. The random variables α and θ have non-informative prior distribution with respective prior pdfs

$$g_1(\alpha) = \frac{1}{c} \quad ; \quad 0 < \alpha < c \tag{21}$$

and

$$g_2(\theta) = \frac{1}{\theta} \quad ; \quad \theta > 0 \tag{22}$$

where as p has Beta distribution of first kind with known parameters a, b. The prior pdf of p is given by

$$g_3(p) = \frac{1}{B(a,b)} p^{a-1} (1-p)^{b-1}; \quad 0 < p < 1, \quad a > 0, \quad b > 0 \tag{23}$$

Based on the assumptions stated above, the joint prior pdf of α, θ and p is

$$g(\alpha, \theta, p) = g_1(\alpha) g_2(\theta) g_3(p) \quad ; \quad 0 < \alpha < c, \quad \theta > 0, \quad 0 < p < 1 \tag{24}$$

Combining the priors given by (21), (22) and (23) with likelihood given by (12), we can easily obtain joint posterior pdf of (α, θ, p) as
$\pi(\alpha, \theta, p|x, r) = \frac{J_1}{J_0}$
where
$J_1 = \alpha^m \theta^{m-1} \prod_{i=1}^m \left[1 - (1+x_i)^{-\alpha}\right]^{\theta-1} \left\{1 - \left[1 - (1+x_i)^{-\alpha}\right]^{\theta}\right\}^{r_i}$
$(1+x_i)^{-(\alpha+1)} p^{\sum_{i=1}^{m-1} r_i + a - 1} (1-p)^{(m-1)(n-m) - \sum_{i=1}^{m-1}(m-i) r_i + b - 1}$

and $J_0 = \int_0^1 \int_0^\infty \int_0^c J_1 d\alpha d\theta dp$

Hence, the respective marginal posterior pdf's of α, θ and p are given by

$$\pi_1(\alpha|x, r) = \int_0^1 \int_0^\infty \frac{J_1}{J_0} d\theta dp \tag{25}$$

$$\pi_2(\theta|x, r) = \int_0^1 \int_0^c \frac{J_1}{J_0} d\alpha dp \tag{26}$$

and

$$\pi_3\left(p|x,r\right)=\int_0^\infty\int_0^c\frac{J_1}{J_0}d\alpha d\theta \tag{27}$$

Usually the Bayes estimators are obtained under Square Error Loss Function (SELF)

$$l_1(\phi,\hat{\phi})=\in_1\left(\phi-\hat{\phi}\right)^2;\quad \in_1>0 \tag{28}$$

Where $\hat{\phi}$ is the estimate of the parameter ϕ and the Bayes estimator $\hat{\phi}_S$ of ϕ comes out to be $E_\phi[\phi]$, where E_ϕ denotes the posterior expectation. This loss function is a symmetric loss function and can only be justified, if over estimation and under estimation of equal magnitude are of equal seriousness. A number of asymmetric loss functions are also available in statistical literature. Let us consider the General Entropy Loss Function (GELF), proposed by Calabria and Pulcini (1996), defined as follows :

$$l_2(\phi,\hat{\phi})=\in_2\left(\left(\frac{\hat{\phi}}{\phi}\right)^\delta-\delta\ln\left(\frac{\hat{\phi}}{\phi}\right)-1\right);\quad \in_2>0 \tag{29}$$

The constant δ, involved in (29), is its shape parameter. It reflects departure from symmetry. When $\delta>0$, it considers over estimation (i.e., positive error) to be more serious than under estimation (i.e., negative error) and converse for $\delta<0$. The Bayes estimator $\hat{\phi}_E$ of ϕ under GELF is given by,

$$\hat{\phi}_E=\left[E_\phi\left(\phi^{-\delta}\right)\right]^{\left(-\frac{1}{\delta}\right)} \tag{30}$$

provided the posterior expectation exits. It may be noted here that for $\delta=-1$, the Bayes estimator under loss (28) coincides with the Bayes estimator under SELF l_1. Expressions for the Bayes estimators $\hat{\alpha}_E$, $\hat{\theta}_E$ and $\hat{p}_E$ for α, θ and p respectively under GELF can be given as

$$\hat{\alpha_E}=\left[\int_0^c\alpha^{-\delta}\pi_1\left(\alpha|x,r\right)d\alpha\right]^{\left(-\frac{1}{\delta}\right)} \tag{31}$$

$$\hat{\theta_E}=\left[\int_0^\infty\theta^{-\delta}\pi_1\left(\theta|x,r\right)d\theta\right]^{\left(-\frac{1}{\delta}\right)} \tag{32}$$

and

$$\hat{p_E}=\left[\int_0^1 p^{-\delta}\pi_3\left(p|x,r\right)dp\right]^{\left(-\frac{1}{\delta}\right)} \tag{33}$$

Substituting the posterior pdfs from (25), (26) and (27) in (31), (32) and (33) respectively and then simplifying, we get the Bayes Estimators $\hat{\alpha}_E,\hat{\theta}_E$ and $\hat{p}_E$ of α,θ and p as follows :

$$\hat{\alpha}_E=\left[\frac{\int_0^\infty\theta^{m-1}\int_0^c\alpha^{m-\delta}\prod_{i=1}^m\left[1-(1+x_i)^{-\alpha}\right]^{\theta-1}\left\{1-\left[1-(1+x_i)^{-\alpha}\right]^\theta\right\}^{r_i}(1+x_i)^{-(\alpha+1)}d\alpha d\theta}{\int_0^\infty\theta^{m-1}\int_0^c\alpha^{m}\prod_{i=1}^m\left[1-(1+x_i)^{-\alpha}\right]^{\theta-1}\left\{1-\left[1-(1+x_i)^{-\alpha}\right]^\theta\right\}^{r_i}(1+x_i)^{-(\alpha+1)}d\alpha d\theta}\right]^{-1/\delta}, \tag{34}$$

$$\hat{\theta}_E = \left[\frac{\int_0^\infty \theta^{m-\delta-1} \int_0^c \alpha^m \prod_{i=1}^m \left[1-(1+x_i)^{-\alpha}\right]^{\theta-1} \left\{1-\left[1-(1+x_i)^{-\alpha}\right]^\theta\right\}^{r_i} (1+x_i)^{-(\alpha+1)} d\alpha d\theta}{\int_0^\infty \theta^{m-1} \int_0^c \alpha^m \prod_{i=1}^m \left[1-(1+x_i)^{-\alpha}\right]^{\theta-1} \left\{1-\left[1-(1+x_i)^{-\alpha}\right]^\theta\right\}^{r_i} (1+x_i)^{-(\alpha+1)} d\alpha d\theta} \right]^{-1/\delta}, \tag{35}$$

and

$$\hat{p}_E = \left[\frac{B(\sum_{i=1}^{m-1} r_i + a - \delta, (m-1)(n-m) - \sum_{i=1}^{m-1}(m-i)r_i + b)}{B(\sum_{i=1}^{m-1} r_i + a, (m-1)(n-m) - \sum_{i=1}^{m-1}(m-i)r_i + b)} \right]^{-1/\delta}. \tag{36}$$

It may be noted that the integrals involved in the expressions for the Bayes estimators $\hat{\alpha}_E$ and $\hat{\theta}_E$ are not reducible in nice closed form. Therefore, we propose the use of numerical integration method for obtaining the estimates. We have used Gauss-quadrature formula for evaluating the inner integral which has finite range and for evaluating the outer integral having range $(0, \infty)$, we used Gauss-Laguerre formula. The evaluations were done by developing program in R-language.

4 Algorithm to Simulate Progressive Type-II Censored Sample With Binomial Removal

For the study of behavior of the estimators obtained in previous sections, we need to simulate progressive type-II censored samples with Binomial removals from specified EPD. To get such a sample, we propose the use of following algorithm :

I. Specify the value of n.

II. Specify the value of m.

III. Specify the value of parameters α, θ and p.

IV. Generate a random sample (S_r) of size n from $EP(\alpha, \theta)$.

V. Generate random number r_i from $\mathrm{B}\left(n - m - \sum_{l=0}^{i-1} r_l, p\right)$, at i^{th} stage for $i = 1, 2, 3, \cdots, m-1$. $(r_0 = 0)$

VI. Get ordered sample S_o from S_r to choose the minimum which will be first observation in desired progressive type-II censored sample S_p.

VII. Drop the observation selected at VI from S_r to have a random sample ${S_r}^*$ of size n^* (less 1 than that of S_r).

VIII. Generate r_i integers (at i^{th} stage) between 1 to n^* and observations corresponding to these numbers are dropped from ${S_r}^*$ to have a random sample ${S_r}^{**}$ of size $n^{**} = n^* - r_i$ and re-designate the random sample in hand as new random sample S_r.

IX. Repeat steps V to VIII $(m-1)$ times.

X. Set r_m according to the following relation.

$$r_m = \begin{cases} n - m - \sum_{l=1}^{m-1} r_l & \text{if } n - m - \sum_{l=1}^{m-1} r_l > 0 \\ 0 & \text{otherwise} \end{cases}$$

and discard all the remaining r_m observations.

5 A Numerical Illustration

For illustration of the proposed estimation procedures given in sections 2 and 3, we have considered below a simulated progressive type-II censored sample with Binomial removals from exponentiated Pareto distribution with $p = .3$, $m = 14$, $\alpha = 2$, and $\theta = .5$, using the above algorithm. The sample thus obtained is given below :

(0.002041782,2), (0.062274706,1), (0.12238516,0), (0.155058203,0), (0.264017913,0), (0.276648679,0), (0.388598753,1), (0.413575829,1), (0.482304597,0), (0.912576221,0), (1.103519893,0), (1.331501842,1), (1.37502576,0), (1.385353469,0).

Substituting value of (x_i, r_i) $i = 1, \cdots, 14$ in (16) and (17), we obtain the normal equations which are solved simultaneously using fixed point iteration procedure to get the MLE's of α and θ. The MLE of p is obtained from (20). The Bayes Estimates are obtained from (36), using the procedure explained in section 3. The results are summarized in table 1. It may be noted from the table that for the sample in hand, all the estimates are larger than the corresponding true values. However the Bayes estimates $\hat{\alpha}_E, \hat{\theta}_E$ and $\hat{p}_E$ with $\delta = 1.5$ is nearer to the true value as compared to other estimates. For $\delta = -1.5$, the Bayes estimate $\hat{\alpha}_S$, $\hat{p}_S$ and MLE $\hat{\theta}_M$ are nearer to the true as compared to other estimates. But on the basis of this particular sample, it will be illogical to conclude that under GELF, we should use $\hat{\alpha}_E$, $\hat{\theta}_E$ and $\hat{p}_E$ if over estimation is more serious than under estimation for their long run use. Therefore, we study the behavior of risks of the estimators in the next section.

Table 1: MLE and Bayes estimate of α, θ and p under progressive type-II censored data with random removal.

δ	α			θ			p		
	$\hat{\alpha}_M$	$\hat{\alpha}_S$	$\hat{\alpha}_E$	$\hat{\theta}_M$	$\hat{\theta}_S$	$\hat{\theta}_E$	$\hat{p}_M$	$\hat{p}_S$	$\hat{p}_E$
1.5	2.3947	2.5492	2.2547	1.0681	1.092	1.0019	0.5455	0.4211	0.3799
-1.5	2.3947	2.3492	2.6038	1.0681	1.092	1.1144	0.5455	0.4211	0.4283

6 Simulation Studies

The estimators $\hat{\alpha}_M, \hat{\theta}_M$and $\hat{p}_M$ denote the MLE's of the parameters α, θ and p respectively while $\hat{\alpha}_S, \hat{\theta}_S$ and $\hat{p}_S$ are corresponding Bayes estimators under SELF and $\hat{\alpha}_E, \hat{\theta}_E$ and $\hat{p}_E$ are the corresponding Bayes estimators under GELF. We compare the estimators obtained under GELF with corresponding MLE's and Bayes estimators under SELF. The comparisons are based on the simulated risks (average loss over sample space) under GELF.

It may be mentioned here that the exact expressions for the risks can not be obtained because estimators are not in nice closed form. Therefore, the risks of the estimators are estimated on the basis of Monte-carlo simulation study of 1000 samples. It may be noted that the risks of the estimators will depend on values of $n, m, \theta, \alpha, p, c, \delta, a$ and b. In order to consider variation in the values of these, we have obtained the simulated risks for $n = 20\,[10]\,40$, $m = 12\,[2]\,18$,$c = 4\,[1]\,8, a = 2\,[2]\,8$, $b = 4\,[2]\,10$, $\theta = 0.5\,[0.5]\,2$, $\alpha = 0.5\,[0.5]\,2.5$, $\delta = \pm 1.5$ and $p = 0.3\,[0.1]\,0.6$.
Generating the progressive sample as mentioned in section 4, the simulated risks under GELF have been obtained for selected values of $n, m, \theta, \alpha, p, b, a, \delta$ and c. The results are summarized in (tables $(2 - 15)$).

6.1 Discussion of the results

It is interesting to note that as n increases, keeping the effective sample size m fixed, the risks of all the estimators increase, in general, except for $\hat{\alpha}_M$ and $\hat{p}_M$ (see tables 2 and 8). On other hand, if the effective sample size m increases for given sample size n, the risks of the estimators of α and θ decrease as m increases, in general, except for the estimators $\hat{\alpha}_M$ and $\hat{\alpha}_S$ (when $\delta = 1.5$). We have noticed that the risks of all the estimators of p increase, often, as m increases (see tables 3 and 9). Due to the increase in the value of θ, the risks of the estimators of α and θ, in general, first decrease then increase. But the risks of the estimators of p do not follow a trend (see table 4 and 10). Further, for the variation in the value of parameter α (when $\delta = 1.5$), we observe that risks of all estimators of α decrease as α increases but risk of estimators of θ and p do not follow a definite trend (see table 5). On the other hand, for $\delta = -1.5$, risks of the estimator $\hat{\alpha}_S$ and $\hat{\alpha}_E$ decrease as α increases but for the rest of the estimators no definite trend is observed (see table 11). The risk of estimators of p (when $\delta = 1.5$) often decrease as p increases and the risk of estimators $\hat{\alpha}_M$ and $\hat{\theta}_M$ increases as p increases but no definite trend for the risks of rest of the estimators of α and θ is noted as p increases (see table 6). For $\delta = -1.5$, risk of estimators of θ and p increases as p increases, but the risks of estimators of α first decrease then increase when p increases (see table 12).

For the variation of the prior parameter c, when $\delta = 1.5$, we observe that no definite trend is followed by the risks of the estimators of α, θ and p except for $\hat{\alpha}_S$ which increases as c increases (see table 7). But, for $\delta = -1.5$, the risk of estimators of α increase as c increases and for rest of estimators no definite trend is observed (see table 13).

It is worth while to remark here that a variation in the values of hyper parameters a and b do not effect the risks of the estimators of α and θ. It is further seen that for variation in hyper parameter a, the risks of estimators of p do not have a definite trend, except for the risk of $\hat{p}_S$ (when $\delta = 1.5$) and $\hat{p}_E$ (when $\delta = -1.5$) which increase as a increases (see table 14). For the variation of parameter b, the risk of estimators of p, except for $\hat{p}_M$ (for $\delta = 1.5$) and $\hat{p}_S$ (for $\delta = -1.5$) decrease for increase in the value of b in the beginning and then increase for further increase in the value of b (see table 15).

It may further be noted from the tables that for $\delta = 1.5$, the risk of $\hat{\theta}_E$ is smaller than the risks of $\hat{\theta}_M$ and $\hat{\theta}_S$. However, for $\delta = -1.5$, the risk of $\hat{\theta}_M$ is smaller than those of $\hat{\theta}_E$ and $\hat{\theta}_S$. Among the estimator of α, $\hat{\alpha}_E$ has smaller risks than those of others for $\delta = -1.5$, but for $\delta = 1.5$, none of the estimators $\hat{\alpha}_M$, $\hat{\alpha}_S$ and $\hat{\alpha}_E$ outperforms the other although there is little variation in the risk. The risk of $\hat{p}_E$ when $\delta = -1.5$ whereas $\hat{p}_S$ when $\delta = 1.5$ is smaller than the risks of other estimators of p.

7 Conclusions

It is expected that the estimator obtained under a particular loss function shall in general perform better than the estimators obtained under other loss functions, but this could not be established in the present study. We have seen above that risk under GELF for the estimators $\hat{\alpha}_E$, $\hat{\theta}_E$ and $\hat{p}_E$ is not always less than those of $\hat{\alpha}_M$, $\hat{\theta}_M$, $\hat{p}_M$, $\hat{\alpha}_S$, $\hat{\theta}_S$, $\hat{p}_S$. Although the risks associated with $\hat{\theta}_E$ is smaller than the risk associated with other estimators in most of the cases. The risks associated with $\hat{\alpha}_E$ and $\hat{p}_E$ are smaller than those of other estimators when $\delta = -1.5$. On other hand if $\delta = 1.5$ risk associated with $\hat{\alpha}_M$ and $\hat{p}_S$ are noted to be smaller than other estimators. Therefore, for the use of the proposed estimators, we may recommend the following:

1. $\hat{\theta}_E$ may be used as an estimator of θ when over estimation is more serious than under estimation. On the other hand, if under estimation is more serious than over estimation, $\hat{\theta}_M$ may be used.

2. $\hat{\alpha}_E$ may be used as an estimator of α when under estimation is more serious than over estimation. Otherwise, $\hat{\alpha}_M$ may be used.

3. $\hat{p}_E$ may be used an estimator of p when under estimation is more serious than over estimation; Otherwise, $\hat{p}_S$ may be used.

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Table 2: Risk of Estimators of α, θ and p under GELF for fixed, m = 14, $\alpha = 2$, $\theta = 0.5$, $p = 0.3$, $c = 5$, $a = 2$, $b = 6$, $\delta = 1.5$.

n	α			θ			p		
	$R_E(\hat{\alpha}_M)$	$R_E(\hat{\alpha}_S)$	$R_E(\hat{\alpha}_E)$	$R_E\left(\hat{\theta}_M\right)$	$R_E\left(\hat{\theta}_S\right)$	$R_E\left(\hat{\theta}_E\right)$	$R_E(\hat{p}_M)$	$R_E(\hat{p}_S)$	$R_E(\hat{p}_E)$
20	0.1459191	0.148331	0.1988515	0.2812842	0.3645443	0.2303535	1.297787	0.05848324	0.07549444
30	0.3910645	0.3889102	0.3911926	0.7785958	0.7743881	0.771871	3.248732	3.231744	3.231876
40	0.3932742	0.3932742	0.3932742	0.7887064	0.7887064	0.7887064	3.279847	3.279847	3.279847

Table 3: Risk of Estimators of α, θ and p under GELF for fixed, n = 20, $\alpha = 2$, $\theta = 0.5$, $p = 0.3$, $c = 5$, $a = 2$, $b = 6$, $\delta = 1.5$.

m	α			θ			p		
	$R_E(\hat{\alpha}_M)$	$R_E(\hat{\alpha}_S)$	$R_E(\hat{\alpha}_E)$	$R_E\left(\hat{\theta}_M\right)$	$R_E\left(\hat{\theta}_S\right)$	$R_E\left(\hat{\theta}_E\right)$	$R_E(\hat{p}_M)$	$R_E(\hat{p}_S)$	$R_E(\hat{p}_E)$
12	0.1878715	0.157183	0.2716257	0.2613284	0.4195576	0.2564014	1.030946	0.05980246	0.07412193
14	0.1373194	0.1472565	0.1849168	0.2697733	0.3752902	0.2349265	1.306697	0.06540113	0.08491809
16	0.1280441	0.1487981	0.1536917	0.2231028	0.3193126	0.2053112	1.453691	0.06657092	0.1023703
18	0.1385783	0.1558678	0.1458456	0.1638702	0.2368763	0.1577898	1.61478	0.06775665	0.1702504

Table 4: Risk of Estimators of α, θ and p under GELF for fixed, m = 14, $\alpha = 2$, $n = 20$, $p = 0.3$, $c = 5$, $a = 2$, $b = 6$, $\delta = 1.5$.

θ	α			θ			p		
	$R_E(\hat{\alpha}_M)$	$R_E(\hat{\alpha}_S)$	$R_E(\hat{\alpha}_E)$	$R_E\left(\hat{\theta}_M\right)$	$R_E\left(\hat{\theta}_S\right)$	$R_E\left(\hat{\theta}_E\right)$	$R_E(\hat{p}_M)$	$R_E(\hat{p}_S)$	$R_E(\hat{p}_E)$
0.5	0.1390676	0.1478296	0.1881797	0.2775102	0.3855086	0.2426155	1.301221	0.06329389	0.0806758
1	0.2148063	0.09182065	0.1127921	0.2169895	0.3307302	0.1806352	1.301558	0.0670841	0.08571692
1.5	0.4232116	0.08317024	0.1037508	0.2347091	0.4079166	0.2110936	1.297559	0.06274592	0.07876336
2	0.4479795	0.3132204	0.3176702	0.3618798	0.4132403	0.3544649	2.784513	2.478634	2.483556

Table 5: Risk of Estimators of α, θ and p under GELF for fixed, n = 20, m = 14, $\theta = 0.5$, $p = 0.3$, $c = 5$, $a = 2$, $b = 6$, $\delta = 1.5$.

α	α			θ			p		
	$R_E(\hat{\alpha}_M)$	$R_E(\hat{\alpha}_S)$	$R_E(\hat{\alpha}_E)$	$R_E\left(\hat{\theta}_M\right)$	$R_E\left(\hat{\theta}_S\right)$	$R_E\left(\hat{\theta}_E\right)$	$R_E(\hat{p}_M)$	$R_E(\hat{p}_S)$	$R_E(\hat{p}_E)$
0.5	0.5892752	0.5881838	0.5814836	0.6142454	0.6560499	0.6047511	2.601751	2.179152	2.186224
1	0.1826074	0.1994308	0.2124504	0.2763414	0.4036504	0.2514949	1.31142	0.06497218	0.08507856
1.5	0.1599294	0.1784462	0.2047829	0.2844317	0.4120996	0.2583423	1.30203	0.06375178	0.0809425
2	0.1423826	0.1473353	0.19152	0.2658764	0.3853314	0.2416606	1.314761	0.06488765	0.08264993
2.5	0.1117617	0.1066854	0.168186	0.257236	0.359291	0.2308104	1.309408	0.06693853	0.0870016

Table 6: Risk of Estimators of α, θ and p under GELF for fixed, n = 20, m = 14, $\alpha = 2$, $\theta = 0.5$, $c = 5$, $a = 2$, $b = 6$, $\delta = 1.5$.

p	α			θ			p		
	$R_E(\hat{\alpha}_M)$	$R_E(\hat{\alpha}_S)$	$R_E(\hat{\alpha}_E)$	$R_E\left(\hat{\theta}_M\right)$	$R_E\left(\hat{\theta}_S\right)$	$R_E\left(\hat{\theta}_E\right)$	$R_E(\hat{p}_M)$	$R_E(\hat{p}_S)$	$R_E(\hat{p}_E)$
0.3	0.3963992	0.3918289	0.3923007	0.7858412	0.7881818	0.7874846	3.269148	3.263524	3.263574
0.4	0.5650922	0.3677883	0.3805516	0.908182	0.7768941	0.759903	1.585762	1.428244	1.429113
0.5	0.5917908	0.3361664	0.3548922	0.9507484	0.7888174	0.746422	1.036194	0.6303212	0.6326188
0.6	0.713041	0.348167	0.3589572	1.113673	0.818969	0.7803772	0.6681636	0.3320889	0.3339984

Table 7: Risk of Estimators of α, θ and p under GELF for fixed, n = 20, m = 14, $\alpha = 2$, $\theta = 0.5$, $p = 0.3$, $a = 2$, $b = 6$, $\delta = 1.5$.

c	α			θ			p		
	$R_E(\hat{\alpha}_M)$	$R_E(\hat{\alpha}_S)$	$R_E(\hat{\alpha}_E)$	$R_E\left(\hat{\theta}_M\right)$	$R_E\left(\hat{\theta}_S\right)$	$R_E\left(\hat{\theta}_E\right)$	$R_E(\hat{p}_M)$	$R_E(\hat{p}_S)$	$R_E(\hat{p}_E)$
4	0.133713	0.1071883	0.1742957	0.252952	0.3382173	0.2123965	1.30623	0.06415044	0.0835438
5	0.1418948	0.1522755	0.1910208	0.2535542	0.3714355	0.2322441	1.305162	0.06795876	0.08826163
6	0.1427238	0.1751292	0.1972878	0.299998	0.4186578	0.2684301	1.30711	0.06882673	0.08422973
7	0.136681	0.1847313	0.2088432	0.2654237	0.403864	0.2547928	1.301522	0.0635916	0.08282253
8	0.1357412	0.1856477	0.2067242	0.2600589	0.3967457	0.2459408	1.300425	0.06046348	0.07562106

Table 8: Risk of Estimators of α, θ and p under GELF for fixed, m = 13, $\alpha = 2$, $\theta = 0.5$, $p = 0.3$, $c = 5$, $a = 2$, $b = 6$, $\delta = -1.5$.

n	α			θ			p		
	$R_E(\hat{\alpha}_M)$	$R_E(\hat{\alpha}_S)$	$R_E(\hat{\alpha}_E)$	$R_E\left(\hat{\theta}_M\right)$	$R_E\left(\hat{\theta}_S\right)$	$R_E\left(\hat{\theta}_E\right)$	$R_E(\hat{p}_M)$	$R_E(\hat{p}_S)$	$R_E(\hat{p}_E)$
20	0.2056455	0.1569264	0.1429979	0.143696	0.1966461	0.2090318	5.477509	0.07577261	0.07074239
30	0.8137402	0.7864026	0.7857344	0.3908113	0.3917175	0.3918124	1.007595	0.9627092	0.9626953
40	0.7887064	0.7887064	0.7887064	0.3932742	0.3932742	0.3932742	0.970276	0.970276	0.970276

Table 9: Risk of Estimators of α, θ and p under GELF for fixed, n = 20, $\alpha = 2$, $\theta = 0.5$, $p = 0.3$, $c = 5$, $a = 2$, $b = 6$, $\delta = -1.5$.

m	α			θ			p		
	$R_E(\hat{\alpha}_M)$	$R_E(\hat{\alpha}_S)$	$R_E(\hat{\alpha}_E)$	$R_E\left(\hat{\theta}_M\right)$	$R_E\left(\hat{\theta}_S\right)$	$R_E\left(\hat{\theta}_E\right)$	$R_E(\hat{p}_M)$	$R_E(\hat{p}_S)$	$R_E(\hat{p}_E)$
12	0.4058337	0.2190515	0.1904377	0.150845	0.2106503	0.2246941	3.850492	0.0687333	0.0655247
14	0.2084325	0.1607437	0.1475378	0.1441369	0.1955944	0.2076074	5.442995	0.06789313	0.06376622
16	0.1391414	0.1343914	0.1288445	0.1218889	0.1658354	0.1752068	7.230498	0.0893988	0.08219142
18	0.1306912	0.1317092	0.1292712	0.09058756	0.1275673	0.1336058	9.159985	0.09593564	0.08259461

Table 10: Risk of Estimators of α, θ and p under GELF for fixed, m = 14, $\alpha = 2$, $n = 20$, $p = 0.3$, $c = 5$, $a = 2$, $b = 6$, $\delta = -1.5$.

θ	α			θ			p		
	$R_E(\hat{\alpha}_M)$	$R_E(\hat{\alpha}_S)$	$R_E(\hat{\alpha}_E)$	$R_E\left(\hat{\theta}_M\right)$	$R_E\left(\hat{\theta}_S\right)$	$R_E\left(\hat{\theta}_E\right)$	$R_E(\hat{p}_M)$	$R_E(\hat{p}_S)$	$R_E(\hat{p}_E)$
0.5	0.1761425	0.1572808	0.1456272	0.1536995	0.2046119	0.2164952	5.513191	0.07700706	0.07222737
1	1.470646	0.1039942	0.09782153	0.1067857	0.1765228	0.1917982	5.441861	0.07209523	0.06773793
1.5	1.692799	0.5481985	0.5459216	0.1987279	0.2165787	0.222747	2.552693	0.6545076	0.6535196
2	1.188187	0.723466	0.7229112	0.7325644	0.7359981	0.7376849	1.405152	0.8862815	0.8858203

Table 11: Risk of Estimators of α, θ and p under GELF for fixed, n = 20, m = 14, $\theta = 0.5$, $p = 0.3$, $c = 5$, $a = 2$, $b = 6$, $\delta = -1.5$.

α	α			θ			p		
	$R_E(\hat{\alpha}_M)$	$R_E(\hat{\alpha}_S)$	$R_E(\hat{\alpha}_E)$	$R_E\left(\hat{\theta}_M\right)$	$R_E\left(\hat{\theta}_S\right)$	$R_E\left(\hat{\theta}_E\right)$	$R_E(\hat{p}_M)$	$R_E(\hat{p}_S)$	$R_E(\hat{p}_E)$
0.5	0.2447477	0.2206069	0.2197203	0.2298253	0.2641567	0.2734419	4.032987	0.3541624	0.3512526
1	0.2076529	0.1897896	0.1761434	0.1524109	0.2048875	0.2173689	5.452927	0.08002817	0.0746755
1.5	0.220915	0.1719469	0.160675	0.1620865	0.2156032	0.2285961	5.461925	0.07080225	0.06736088
2	0.1978864	0.1686081	0.1555704	0.1502385	0.2046902	0.2168534	5.504892	0.07940477	0.07453942
2.5	0.1845989	0.1455462	0.1299962	0.143902	0.1938008	0.205096	5.487717	0.06629183	0.06240312

Table 12: Risk of Estimators of α, θ and p under GELF for fixed, n = 20, m = 14, $\alpha = 2$, $\theta = 0.5$, $c = 5$, $a = 2$, $b = 6$, $\delta = -1.5$.

p	α			θ			p		
	$R_E(\hat{\alpha}_M)$	$R_E(\hat{\alpha}_S)$	$R_E(\hat{\alpha}_E)$	$R_E\left(\hat{\theta}_M\right)$	$R_E\left(\hat{\theta}_S\right)$	$R_E\left(\hat{\theta}_E\right)$	$R_E(\hat{p}_M)$	$R_E(\hat{p}_S)$	$R_E(\hat{p}_E)$
0.3	0.1926444	0.1636426	0.1491117	0.1456174	0.1972236	0.2096048	5.492038	0.0639176	0.06043838
0.4	0.1601832	0.1448309	0.136481	0.1785616	0.2330036	0.2468082	9.641454	0.0687014	0.06125533
0.5	0.147586	0.1357172	0.1294271	0.1887159	0.2459813	0.2608011	14.58705	0.09219164	0.08140336
0.6	0.1703985	0.1403303	0.1355625	0.2083287	0.2658582	0.2811275	20.15317	0.1459551	0.1314902

Table 13: Risk of Estimators of α, θ and p under GELF for fixed, n = 20, m = 14, $\alpha = 2$, $\theta = 0.5$, $p = 0.3$, $a = 2$, $b = 6$, $\delta = -1.5$.

c	α			θ			p		
	$R_E(\hat{\alpha}_M)$	$R_E(\hat{\alpha}_S)$	$R_E(\hat{\alpha}_E)$	$R_E\left(\hat{\theta}_M\right)$	$R_E\left(\hat{\theta}_S\right)$	$R_E\left(\hat{\theta}_E\right)$	$R_E(\hat{p}_M)$	$R_E(\hat{p}_S)$	$R_E(\hat{p}_E)$
4	0.1798442	0.135267	0.1208644	0.1508042	0.1891916	0.2005542	5.458144	0.06754857	0.06390081
5	0.1694883	0.1508413	0.1398245	0.1533502	0.2040592	0.2160421	5.493616	0.07057307	0.06637562
6	0.1768365	0.1655576	0.1540674	0.1458951	0.202437	0.2149109	5.492914	0.08049661	0.07514648
7	0.1982377	0.1677997	0.1588099	0.1527374	0.2163435	0.2294159	5.483429	0.07095818	0.0667329
8	0.2015065	0.1978975	0.1834582	0.1585488	0.2157997	0.2289757	5.496052	0.07621357	0.0715007

Table 14: Risk of Estimators of α, θ and p under GELF for fixed, n = 20, m = 14, $\alpha = 2$, $\theta = 0.5$, $p = 0.3$, $c = 5$, $b = 6$, $\delta = \pm 1.5$.

a	$\delta = 1.5$			$\delta = -1.5$		
	$R_E(\hat{p}_M)$	$R_E(\hat{p}_S)$	$R_E(\hat{p}_E)$	$R_E(\hat{p}_M)$	$R_E(\hat{p}_S)$	$R_E(\hat{p}_E)$
2	1.306358	0.06609435	0.08400493	5.398939	0.06622675	0.06232504
4	1.295787	0.08748874	0.06644892	5.459162	0.06402437	0.06633571
6	1.303061	0.1449965	0.1064253	5.429403	0.09412163	0.09857798
8	1.304805	0.2291079	0.1793196	5.45997	0.1275746	0.1326883

Table 15: Risk of Estimators of α, θ and p under GELF for fixed, n = 20, m = 14, $\alpha = 2$, $\theta = 0.5$, $p = 0.3$, $c = 5$, $a = 2$, $\delta = \pm 1.5$.

b	$\delta = 1.5$			$\delta = -1.5$		
	$R_E(\hat{p}_M)$	$R_E(\hat{p}_S)$	$R_E(\hat{p}_E)$	$R_E(\hat{p}_M)$	$R_E(\hat{p}_S)$	$R_E(\hat{p}_E)$
4	1.308718	0.08872191	0.08782098	5.451559	0.0767966	0.07589028
6	1.307904	0.06696796	0.08681964	5.513005	0.07914773	0.07417736
8	1.313298	0.05896052	0.09528147	5.499888	0.0865787	0.07742965
10	1.309155	0.06726842	0.1204681	5.484824	0.112021	0.09878106

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