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RELIABILITY COMPARISON OF SYSTEMS OF DIFFERENT ORDERS USING PSEUDO-SIGNATURES

Soma Roychowdhury*

Indian Institute of Social Welfare and Business Management, Calcutta, India

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Abstract: *Comparison among a number of systems of same or different orders is a very common issue to the reliability practitioners. Various methods and criteria for the said purpose are available in the literature, but are not applicable to all system designs. The signature of a system with independent and identically distributed component lifetimes has been found to be very useful in the study and comparison of engineered systems. But the signatures and tail probability vectors have been used for comparing the systems of same order. The method fails if the systems are of different orders. This paper discusses an approach to comparing the systems of different orders by introducing a new distribution-free measure. Even if the component reliabilities are not known or the reliability function is difficult to evaluate, this method shows a way to compare the reliabilities of the systems in order to select the best one for which reliability is maximum.*

Keywords: *Reliability function, structure function, system signature, system reliability.*

1. Introduction

In reliability engineering, comparison of a number of systems of same or different orders and selection of the best design from them are of great concern. Though a number of methods are available for comparing the systems, but they are not applicable to all types of engineered systems. Earlier works have considered the comparison among the systems of same order. While making comparison we can use the direct method of comparison of system reliabilities and the indirect method of comparison via signatures of the systems. It is easy to see that the comparison among systems of different orders is possible by direct method when the systems are simple and

* E-mail: srcdb@yahoo.com

do not involve many components, but it is almost impossible to use the direct method for comparing complex systems, as then the analytical difficulties in computing the values of a particular measure of system performance make the comparison almost impossible. In another approach (Kochar et al., 1999) the performance of the competing systems can be compared by finding the signatures of the systems and comparing the tail probabilities in the respective signature vectors. Kochar et al.(1999) compared systems of same order when the components are assumed to have independent and identically distributed lifetimes. Roychowdhury and Bhattacharya(2009) used system signatures to compare the performance of several systems, where component lives are independent, but not necessarily identical. Again the method using signature and the dominance of the tail probability vectors of the respective signature vectors is difficult to apply if the systems are not of same order. In such cases the signature vectors under comparison belong to different vector spaces and hence not comparable directly. This paper discusses an approach to comparing the systems of different orders by introducing a new distribution-free measure. Even if the component reliabilities are not known or the reliability function is difficult to evaluate, the method discussed here shows a direction to compare the reliabilities of the systems of different orders in order to select the best one with the help of pseudo-signatures.

The organization of this paper is as follows: the present section introduces the work, section 2 discusses the definitions and notation necessary for discussing the results of this paper, the main results are discussed in section 3, and section 4 finally gives an important discussion and conclusion.

2. Definitions and Notation

Let us consider a system of n independent components with component reliability r_i at time t, i = 1, 2, ..., n. A system can be considered to be a collection of minimal cut sets (a minimal set of components whose failure guarantees system failure) and its life can be represented in terms of the lives of the minimal cut sets (Barlow and Proschan (1981)). Let X_1, X_2 , ..., X_n be the component lifetimes, and let T be the lifetime of the system which is defined as:

$$T = \min_{1 \le i \le m} \max_{j \in K_i} X_j , \qquad (1)$$

where K_1, K_2, …, K_m are the minimal cut sets of the system under consideration. We may compare several systems directly by computing their lives using (1) or we may determine $P(T > t)$ of the competing systems. But this direct comparison procedure becomes complicated when the number of components in a system gets larger. Let us now define the signature of an n-component system. The signature of a coherent system of order n is defined as the probability distribution:

$\mathbf{p} = (p_1, p_2, ..., p_n)$,

for which

$p_i = P(T = X_{(i)})$, $i = 1, 2, \ldots, n$

and

$\sum p_i = 1$,

where $X_{(i)}$ is the i^{th} order statistic of the component lifetimes, $i = 1, 2, \ldots, n$. It is to be noted that the signature **p** depends only on the system design and not on the underlying continuous distribution of the component lifetimes. Thus, the probability that the i^{th} component-failure will kill the system is solely dependent on the likelihood that the last functioning component in a minimal cut set is the i^{th} component to fail overall.

Consider an n-component coherent system. Note that we can associate a probability vector $\mathbf{p}^{n\times 1}$ with every coherent system of order n and for a given structural design, which is referred to as the system's signature. In relation to the structural design of the system and ordering of the component lifetimes $X_1, X_2, ..., X_n$, the vector **p** is a very important quantity, and the system's performance can be measured using the signature vector **p**.

The signature vector **p** can be defined as the probability vector with elements:

$$p_i = \frac{\text{number of orderings for which the } i^{\text{th}} \text{ failure kills the system}}{n!}, \; i = 1, 2, ..., n.$$

The tail probability vector of the vector **p** can be written as:

$\mathbf{v} = (v_1, v_2, \ldots, v_n)$,

where $v_j = \sum_{i=j}^{n} p_i$.

For two signature vectors, $\mathbf{p_1}$ and $\mathbf{p_2}$, corresponding to two systems of order n, we write:

$\mathbf{p_1} \leq^{st} \mathbf{p_2}$

if and only if

$v_{1j} \leq v_{2j}$ or $\sum_{i=j}^{n} p_{1i} \leq \sum_{i=j}^{n} p_{2i}$, $j = 1, 2, \ldots,$

where $\mathbf{v}_i = (v_{i1}, v_{i2}, \ldots, v_{in})$, with $v_{ij} = \sum_{k=j}^{n} p_{ik}$, $j = 1, 2, \ldots, n$, is the tail probability vector of the system whose signature vector is $\mathbf{p}_i$, $i = 1, 2$. System reliability can be represented, as follows, using signatures:

$$P(T > t) = \sum_{i=1}^{n} p_i P(X_{(i)} > t),$$

and hence expected system life can be expressed as:

$$E(T) = \sum_{i=1}^{n} p_i E(X_{(i)}) .$$

Relation between stochastically ordered signatures and system lifetimes is given in Kochar et al.(1999). Now we define a structure function and a reliability function of a system.
Let the state of the i^{th} component be denoted by a binary variable, u_i such that:

u_i = 0, if i^{th} component is in failing state (at time t)
= 1, if i^{th} component is in functioning state (at time t).

For an n-component system, the state vector of the components is given by:

$\mathbf{u} = (u_1, u_2, ..., u_n)$.

Let r_i be the probability that the i^{th} component is functioning (at time t), i.e., $r_i = P(u_i = 1)$. Note that for notational simplicity, we, without loss of generality, suppress the time variable t in all notation. Let the vector of the component reliabilities be:

$\mathbf{r} = (r_1, r_2, ..., r_n)$.

Here

$E(u_i) = 1.P(u_i = 1) + 0.P(u_i = 0) = 1. r_i + 0.(1 - r_i) = r_i$.

Let the state of the system be given by a function, called structure function, denoted by $\phi(\mathbf{u}) = \phi(u_1, u_2, ..., u_n)$, where:

$\phi(\mathbf{u})$ = 0, if the system is not functioning
= 1, if the system is functioning.

$\phi(\mathbf{u})$ can be expressed in terms of minimal cut sets. Suppose, $K_1, K_2, ..., K_m$ be the m minimal cut sets of a system. Let us define a function $\zeta_j = 1 - \prod_{i \in K_j}(1 - u_i)$, $j = 1, 2, \ldots, m$. Then:

$$\phi(\mathbf{u}) = \prod_{j=1}^{m} \zeta_j = \prod_{j=1}^{m} \{1 - \prod_{i \in K_j}(1 - u_i)\} . \quad (2)$$

From the structure function of a system, its reliability function can be determined as:

$$h(\mathbf{r}) = E[\phi(\mathbf{u})]. \quad (3)$$

In $\phi(\mathbf{u}) = \phi(u_1, u_2, ..., u_n)$, if $u_i = 1$, we write $\phi(\mathbf{u})$ as $\phi(1_i, \mathbf{u})$, and if $u_i = 0$, we write $\phi(\mathbf{u})$ as $\phi(0_i, \mathbf{u})$. In other words, $\phi(1_i, \mathbf{u})$ denotes a state vector, indicating that the state of the i^{th} component is 1, i.e., i^{th} component is functioning, and $\phi(0_i, \mathbf{u})$ denotes a state vector, indicating that the state of i^{th} component is 0, i.e., the i^{th} component is not functioning. Note that for an irrelevant component, $\phi(1_i, \mathbf{u}) = \phi(0_i, \mathbf{u})$, for all ($i$, $\mathbf{u}$).

3. Main Results

Here we discuss how system signatures and tail probability vectors can be used to compare system lives of a number of competing systems of different orders. If we wish to compare two systems with same number of components having i.i.d. lifetimes, the system having stochastically larger signature is considered to be superior, as it has stochastically longer life. But this works for the systems with same number of components only. If the systems are of different number of components, we can still compare them by adding some dummy components anywhere in parallel to any of the components in the lower order system to make the competing systems have equal number of components. For the purpose of comparison we introduce a distribution-free measure, called 'pseudo-signature' of an augmented system that is obtained by adding the dummy components. These dummy components are failed components having zero reliability so that their presence in the system design (added in parallel) does not cause deterioration of the system reliability. The dummy component (represented by 'd' in the system designs) can be put in any position in the system design, but that has to be in parallel, as described in Figures 1-3. Note that in terms of system life or system reliability, the augmented system is an equivalent form of the original system.

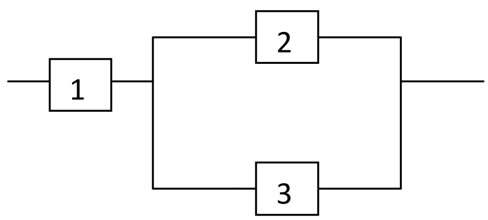

Figure 1. System A (3-component system).

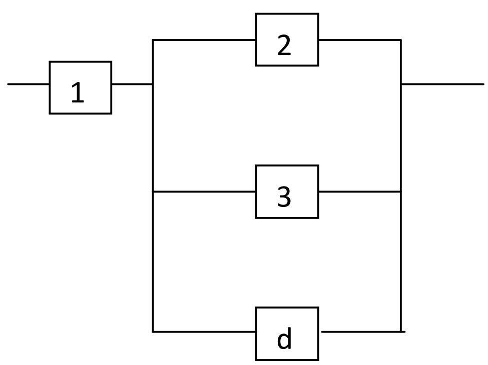

Figure 2. System A (after adding a dummy component in parallel to components 2 and 3).

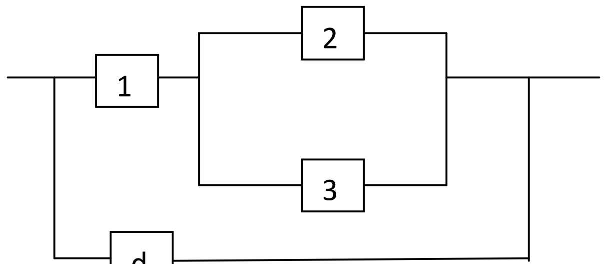

Figure 3. System A (after adding a dummy component in parallel to component 1).

Let us now define the pseudo-signature of an augmented system of order n, which includes m dummy components. It is defined as the probability distribution

$$\mathbf{p}^* = (p^*_1, p^*_2, \dots, p^*_n),$$

where p^*_i is the probability that the i^{th} component-failure will kill the system, $i = 1, 2, \dots, n$, with $\sum p^*_i = 1$. The pseudo-signature vector $\mathbf{p}^*$ can be defined as the probability vector with elements:

$$p^*_i = \frac{\text{number of possible orderings of component lives for which the } i^{th} \text{ failure kills the system}}{(n-m)!}, \qquad (4)$$

with $i = 1, 2, \dots, n$

where m is the number of dummy components added to the system to augment it to order n, because the number of all possible orders of component lives $X_1, X_2, \dots, X_n$ will then be $(n\text{-}m)!$ only. It is to be noted that the first m elements of the pseudo-signature vector $\mathbf{p}^*$ are always zero, (i.e., $p^*_i = 0$, $i = 1, 2, \dots, m$), when there are m dummy components in the augmented system of order n. Note that if $m = 0$, i.e., no dummy is added, then the pseudo-signature becomes same as the signature of the system. In other words, for a system design with no dummy components, there is no difference between its pseudo-signature and the signature vectors.
The tail probability vector of $\mathbf{p}^*$ can be written as:

$\mathbf{v} = (v_1, v_2, \dots, v_n)$,

where $v_j = \sum_{i=j}^{n} p_i^*$.

By using the concept of stochastic ordering of two probability vectors, two pseudo-signature vectors, $\mathbf{p}^*_1$ and $\mathbf{p}^*_2$, corresponding to two systems of order n, can be ordered as follows:

$\mathbf{p}^*_1 \leq^{st} \mathbf{p}^*_2$

if and only if $v_{1j} \leq v_{2j}$ or $\sum_{i=j}^{n} p_{1i}^* \leq \sum_{i=j}^{n} p_{2i}^*$, $j = 1, 2, \dots,$

where $\mathbf{v}_i = (v_{i1}, v_{i2}, \ldots, v_{in})$, with $v_{ij} = \sum_{k=j}^{n} p^*_{ik}$, $j = 1, 2, \ldots, n$, is the tail probability vector of the system whose pseudo-signature vector is $\mathbf{p}^*_i$, $i = 1, 2$. In the context of stochastic ordering of random vectors the work of Shanthikumar (1987) may be referred to.

An example to describe how to obtain the pseudo-signature vector of a system will be helpful in this context. For System A with one dummy component (Figure 2 or 3), suppose the component lives be X_1, X_2 , X_3, X_4 , where X_4 is the life of the dummy component. All possible orders of the component lives are tabulated below. Hence, from Table 1, we get $p^*_1 = 0$, $p^*_2 = 2/6 = 1/3$, $p^*_3 = 4/6 = 2/3$ and $p^*_4 = 0$, i.e., the pseudo-signature vector of System A, $\mathbf{p}^* = (0, 1/3, 2/3, 0)$. Its tail probability vector is $\mathbf{v} = (1, 1, 2/3, 0)$.

Table 1. Table for calculation of pseudo-signature vector of System A.

Possible order of component lifetimes	Life of the component that kills the system (System life)	i^* (i^{th} failure that kills the system)
$X_4 < X_1 < X_2 < X_3$	X_1	2
$X_4 < X_2 < X_1 < X_3$	X_1	3
$X_4 < X_3 < X_2 < X_1$	X_2	3
$X_4 < X_1 < X_3 < X_2$	X_1	2
$X_4 < X_2 < X_3 < X_1$	X_3	3
$X_4 < X_3 < X_1 < X_2$	X_1	3
**in the ordered arrangement of component lives, position of the component that kills the system*		

Note that it is not always easy to obtain the reliability of all types of systems. Therefore we need to look for a method which is capable of ranking the systems in respect of their reliabilities, without calculating the reliabilities. The pseudo-signatures of the competing systems can be made use of for this purpose.

Now we prove the following results which show the way to compare the system designs having different number of components via their pseudo-signatures.

Result 1. Consider System A and System B, as pictured in Figures 1 and 4, respectively. If the lower order system, i.e., System A here, is augmented by adding a dummy component in parallel to any of its components to make the two systems of same order, then:

$$\mathbf{p}^*_{\mathbf{B}} <^{st} \mathbf{p}^*_{\mathbf{A}}$$

and

$$h(\mathbf{B}) < h(\mathbf{A}),$$

where $\mathbf{p}^*_{\mathbf{A}}$ and $\mathbf{p}^*_{\mathbf{B}}$ are the respective pseudo-signature vectors of the augmented System A and System B, and $h(\mathbf{A})$, $h(\mathbf{B})$ are the respective system reliabilities.

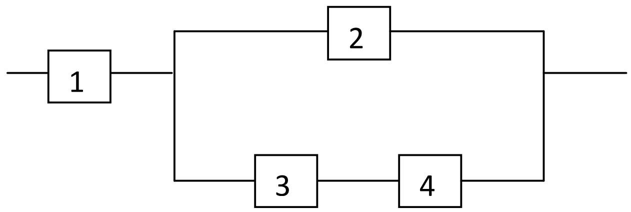

Figure 4. System B.

Proof. The pseudo-signature vector of augmented System A (after adding a dummy component) is determined as $\mathbf{p}^*_{\mathbf{A}}$ = (0, 1/3, 2/3, 0) and the signature vector or the pseudo-signature vector of System B is obtained as $\mathbf{p}^*_{\mathbf{B}}$ = (1/4, 7/12, 1/6, 0), using (4).
Let $\mathbf{v_A}$ and $\mathbf{v_B}$ be the respective tail probability vectors of the pseudo-signature vectors of the two systems. Here we have $\mathbf{v_A}$ = (1, 1, 2/3, 0) and $\mathbf{v_B}$ = (1, 3/4, 1/6, 0).
Hence $\mathbf{v_A}$ dominates $\mathbf{v_B}$, which implies that $\mathbf{p}^*_{\mathbf{B}} <^{st} \mathbf{p}^*_{\mathbf{A}}$.
Now we show that the reliability of System A is higher than that of System B.
Let r_i be the component reliability of the i^{th} component, i = 1, 2, 3, 4.
The minimal cutsets of System A are {1}, {2, 3}.
Then from (2) and (3), its reliability function of System A will be:

$$h(\mathbf{A}) = r_1\,(r_2 + r_3 - r_2 r_3). \tag{5}$$

The minimal cutsets of System B are {1}, {2, 3}, {2, 4}.
Hence, using (2) and (3), we have the reliability function of System B as:

$$h(\mathbf{B}) = r_1\,(r_2 + r_3 - r_2 r_3)\,(r_2 + r_4 - r_2 r_4) \tag{6}$$

From (5) and (6) we can write:

$h(\mathbf{B}) < h(\mathbf{A})$,

since $(r_2 + r_4 - r_2 r_4)$ is the probability that the component 2 or 4 fails.
Thus we can conclude that System A having stochastically larger pseudo-signature is better than System B in the sense of having higher reliability. Now let us consider the following systems of different orders:

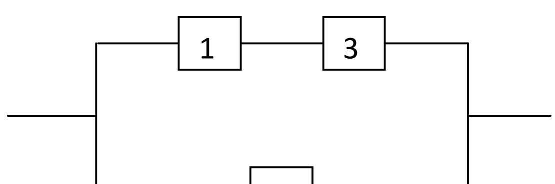

Figure 5. System C (3-component system).

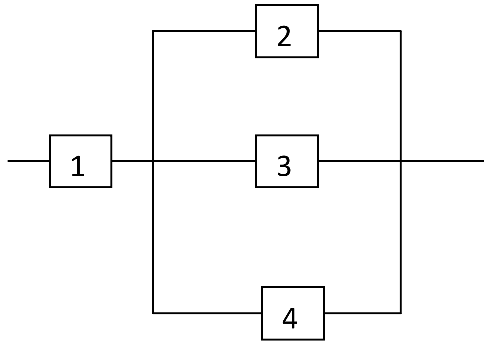

Figure 6. System D (4-component system).

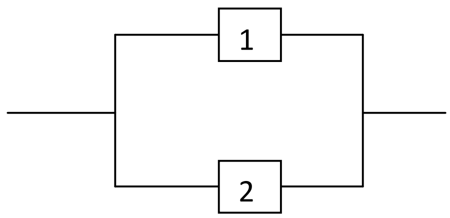

Figure 7. System E (2-component system).

Result 2. Consider System A and System D, as pictured in Figures 1 and 6, respectively. If the lower order system, i.e., System A here, is augmented by adding a dummy component in parallel to any of its components to make the two systems of same order, then:

$$\mathbf{p}^*_{\mathbf{A}} <^{st} \mathbf{p}^*_{\mathbf{D}}$$

and

$$h(\mathbf{A}) < h(\mathbf{D}),$$

where $\mathbf{p}^*_{\mathbf{A}}$ and $\mathbf{p}^*_{\mathbf{D}}$ are the respective pseudo-signature vectors of the augmented System A and System D, and $h(\mathbf{A})$, $h(\mathbf{D})$ are the respective system reliabilities.

Proof. The pseudo-signature vector of augmented System A is given by $\mathbf{p}^*_{\mathbf{A}}$ = (0, 1/3, 2/3, 0) and the signature vector or the pseudo-signature vector of System D is $\mathbf{p}^*_{\mathbf{D}}$ = = (1/4, 1/4, 1/2, 0). Let $\mathbf{v_A}$ and $\mathbf{v_D}$ be the respective tail probability vectors of the two systems. Here we have $\mathbf{v_A}$ = (1, 1, 2/3, 0), and $\mathbf{v_D}$ = (1, 3/4, 1/2, 0). Hence $\mathbf{v_A}$ dominates $\mathbf{v_D}$, which implies that $\mathbf{p}^*_{\mathbf{A}} <^{st} \mathbf{p}^*_{\mathbf{D}}$.
Now we show that the reliability of System D is higher than that of System A.
Let r_i be the component reliability of the i^{th} component, i = 1, 2, 3, 4.
The minimal cut sets of System A are {1}, {2, 3}.
Then from (2) and (3), its reliability function will be:

$$h(\mathbf{A}) = r_1\,(r_2 + r_3 - r_2 r_3). \tag{7}$$

The minimal cut sets of System D are {1}, {2, 3, 4}.
Hence, using (2) and (3), we have the reliability function of System D as:

$$h(\mathbf{p_D}) = r_1\{1-(1-r_2)(1-r_3)(1-r_4)\}$$
$$= r_1\,(r_2 + r_3 + r_4 - r_2 r_3 - r_2 r_4 - r_3 r_4 + r_2 r_3 r_4)$$
$$= r_1\{(r_2 + r_3 - r_2 r_3) + r_4\,(1-r_2) - r_3 r_4\,(1-r_2)\}$$
$$= r_1\{(r_2 + r_3 - r_2 r_3) + r_4\,(1-r_2)\,(1-r_3)\} \qquad (8)$$

From (7) and (8) we can write:

$h(\mathbf{A}) < h(\mathbf{D})$.

Thus we can conclude that System A having stochastically smaller pseudo-signature is better than System D in the sense of having smaller reliability. From the above two theorems we have: ■

$$\text{D} \succ^b \text{A} \succ^b \text{B}, \qquad (9)$$

where '$\succ^b$' indicates that the system on its left is 'better' than the system on its right.
Result 3. Consider System D and System E, as pictured in Figures 6 and 7, respectively. If the lower order system, i.e., System E here, is augmented by adding two dummy components in parallel to any of its components to make the two systems of same order (as in Figure 8), then:

$\mathbf{p}^*_{\mathbf{D}} <^{st} \mathbf{p}^*_{\mathbf{E}}$

and

$h(\mathbf{D}) < h(\mathbf{E})$,

where $\mathbf{p}^*_{\mathbf{D}}$ and $\mathbf{p}^*_{\mathbf{E}}$ be the respective pseudo-signature vectors of System D and the augmented System E, and $h(\mathbf{E})$, $h(\mathbf{D})$ are the respective system reliabilities.

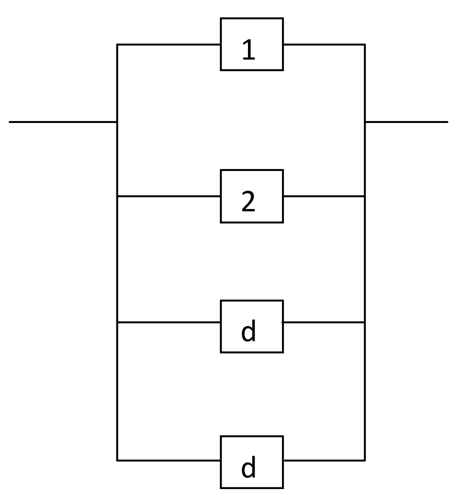

Figure 8. System E (after adding dummy components).

Proof. The pseudo-signature vector of augmented System E (after adding two dummy components) is given by $\mathbf{p}^*_{\mathbf{E}}$ = (0, 0, 0, 1) and the signature vector or the pseudo-signature vector of System D is $\mathbf{p}^*_{\mathbf{D}}$ = (1/4, 1/4, 1/2, 0).
Let $\mathbf{v_D}$ and $\mathbf{v_E}$ be the respective tail probability vectors of the two systems D and E. Here we have $\mathbf{v_E}$ = (1, 1, 1, 1) and $\mathbf{v_D}$ = (1, 3/4, 1/2, 0). Hence $\mathbf{v_E}$ dominates $\mathbf{v_D}$, which implies that $\mathbf{p}^*_{\mathbf{D}} <^{st} \mathbf{p}^*_{\mathbf{E}}$.
Now we show that the reliability of System E is higher than that of System D.
Let r_i be the component reliability of the i^{th} component, $i = 1, 2, 3, 4$.
The minimal cut set of System E is {1, 2}.
Then from (2) and (3), its reliability function will be:

$$h(\mathbf{E}) = r_1 + r_2 - r_1 r_2 = r_1 + r_2(1 - r_1) \tag{10}$$

The minimal cut sets of System D are {1}, {2, 3, 4}.
From (8) we have the reliability function of System D as:

$$h(\mathbf{D}) = r_1\{1 - (1 - r_2)(1 - r_3)(1 - r_4)\} = r_1 - r_1(1 - r_2)(1 - r_3)(1 - r_4) \tag{11}$$

Comparing (10) and (11) we have:

$h(\mathbf{D}) < h(\mathbf{E})$.

Thus we can conclude that System E having stochastically larger signature is better than System D in the sense of having higher reliability.
From Theorems 1, 2 and 3 we have:

$$\text{E} \succ^b \text{D} \succ^b \text{A} \succ^b \text{B}. \tag{12}$$

Result 4. Consider Systems A, B, C and E, as pictured in Figures 1, 4, 5, 7. They are, respectively, of order 3, 4, 3 and 2. If the lower order systems, i.e., Systems A, C and E here, are augmented respectively by adding one, one and two dummy components in parallel to make all the systems of same order (as in Figures 2, 8, 9), then:

$\mathbf{p}^*_{\mathbf{E}} <^{st} \mathbf{p}^*_{\mathbf{C}} <^{st} \mathbf{p}^*_{\mathbf{A}} <^{st} \mathbf{p}^*_{\mathbf{B}}$

and

$h(\mathbf{B}) < h(\mathbf{A}) < h(\mathbf{C}) < h(\mathbf{E})$,

where $\mathbf{p}^*_{\mathbf{A}}$, $\mathbf{p}^*_{\mathbf{B}}$, $\mathbf{p}^*_{\mathbf{C}}$ and $\mathbf{p}^*_{\mathbf{E}}$ be the respective pseudo-signature vectors of the Systems A, B, C and E, and $h(\mathbf{E})$, $h(\mathbf{C})$, $h(\mathbf{A})$, $h(\mathbf{B})$ are respective system reliabilities.

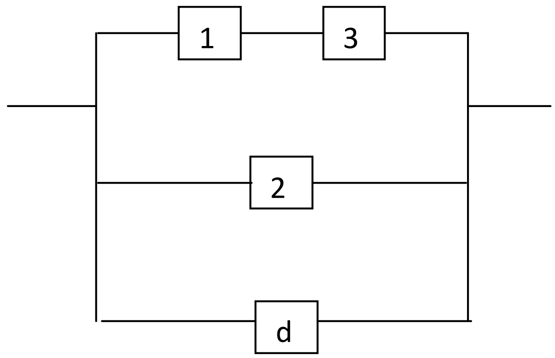

Figure 9. System C (after adding a dummy component).

Proof. The pseudo-signature vectors of the augmented Systems A, C, E (after adding necessary number of dummy components in parallel to make them of same order) are given by $\mathbf{p}^*_{\mathbf{A}}$ = (0, 1/3, 2/3, 0), $\mathbf{p}^*_{\mathbf{C}}$ = (0, 0, 2/3, 1/3), $\mathbf{p}^*_{\mathbf{E}}$ = (0, 0, 0, 1), and the signature or pseudo-signature vector of System B is $\mathbf{p}^*_{\mathbf{B}}$ = (1/4, 7/12, 1/6, 0). The respective tail probability vectors of the systems are: $\mathbf{v}_{\mathbf{A}}$ = (1, 1, 2/3, 0), $\mathbf{v}_{\mathbf{C}}$ = (1, 1, 1, 1/3), $\mathbf{v}_{\mathbf{E}}$ = (1, 1, 1, 1) and $\mathbf{v}_{\mathbf{B}}$ = (1, 3/4, 1/6, 0). Hence $\mathbf{v}_{\mathbf{E}}$ dominates $\mathbf{v}_{\mathbf{A}}$, $\mathbf{v}_{\mathbf{B}}$, $\mathbf{v}_{\mathbf{C}}$, and $\mathbf{v}_{\mathbf{C}}$ dominates $\mathbf{v}_{\mathbf{A}}$, $\mathbf{v}_{\mathbf{B}}$, and $\mathbf{v}_{\mathbf{A}}$ dominates $\mathbf{v}_{\mathbf{B}}$, which implies that $\mathbf{p}^*_{\mathbf{B}} <^{st} \mathbf{p}^*_{\mathbf{A}} <^{st} \mathbf{p}^*_{\mathbf{C}} <^{st} \mathbf{p}^*_{\mathbf{E}}$.
Now we try to determine and order the reliabilities of the Systems A, B, C, and E.
Let r_i be the component reliability of the i^{th} component, i = 1, 2, 3, 4.
The minimal cutsets of System A are {1}, {2, 3}, of System B are {1}, {2, 3}, {2,4}, of System C are {1, 2}, {2, 3} and of System E is {1, 2}.
Then, the reliability function of Systems A, B, C, E will be $h(\mathbf{A}) = r_1\,(r_2 + r_3 - r_2 r_3)$, $h(\mathbf{B}) = r_1\,(r_2 + r_3 - r_2 r_3)\,(r_2 + r_4 - r_2 r_4)$, $h(\text{C}) = (r_1 + r_2 - r_1 r_2)\,(r_2 + r_3 - r_2 r_3)$ and $h(\mathbf{E}) = r_1 + r_2 - r_1 r_2$.
Hence we can write:

$h(\mathbf{B}) < h(\mathbf{A}) < h(\mathbf{C}) < h(\mathbf{E})$.

Thus we can conclude that System E having stochastically largest signature is the best among the given systems of different orders in the sense of having highest reliability, and their orders are as follows:

$$\text{E} \succ^{b} \text{C} \succ^{b} \text{A} \succ^{b} \text{B}. \tag{13}$$

Result 5. Consider Systems C and D, as pictured in Figures 5 and 7, respectively, with independent and identically distributed component lifetimes. If the lower order system, i.e., System C here, is augmented by adding a dummy component in parallel to the components of System C to make the systems of same order, then:

$\mathbf{p}^*_{\mathbf{D}} <^{st} \mathbf{p}^*_{\mathbf{C}}$

and

$h(\mathbf{C}) < h(\mathbf{D})$.

where $\mathbf{p}^*_{\mathbf{C}}$ and $\mathbf{p}^*_{\mathbf{D}}$ are the respective pseudo-signature vectors of the Systems C and D, and $h(\mathbf{C})$, $h(\mathbf{D})$ are respective system reliabilities.

Proof. The pseudo-signature vector of System C (after adding a dummy component) is given by $\mathbf{p}^*_{\mathbf{C}} = (0, 0, 2/3, 1/3)$ and the signature or the pseudo-signature vector of System D is $\mathbf{p}^*_{\mathbf{D}} = (1/4, 1/4, 1/2, 0)$.

The respective tail probability vectors of the systems C and D are as follows: $\mathbf{v_C} = (1, 1, 1, 1/3)$ and $\mathbf{v_D} = (1, 3/4, 1/2, 0)$. Here $\mathbf{v_C}$ dominates $\mathbf{v_D}$, which implies that $\mathbf{p}^*_{\mathbf{D}} <^{st} \mathbf{p}^*_{\mathbf{C}}$.

Now we show that the reliability of System C is higher than that of System D.

Let r be the component reliability.

The reliability functions of Systems C and D, when the components are i.i.d., are:

$$h(\mathbf{C}) = (2r - r^2)^2 = r^2\,(2 - r)^2 = r^2\,(4 - 4r + r^2) = r^2\{(3 - 3r + r^2) + (1 - r)\}$$

$$h(\mathbf{D}) = r\,(3r - 3r^2 + r^3) = r^2(3 - 3r + r^2).$$

Hence we have:

$$h(\mathbf{D}) < h(\mathbf{C}).$$

Thus we can conclude that C $\succ^b$ D, i.e., System C having stochastically larger signature is better than System D in the sense of having higher reliability.

If the component lifetimes are i.i.d., then combining (12), (13) and using Result 5 we can compare the systems A, B, C, D, E and we have:

$$\text{E} \succ^b \text{C} \succ^b \text{D} \succ^b \text{A} \succ^b \text{B}.$$

This way we can make all competing coherent systems of same order by adding necessary number of dummy components in parallel and find the pseudo-signature vectors and tail probability vectors of the systems to rank them in order to compare their reliabilities to select the best one.

4. Conclusion and Discussion

This paper discussed an approach to comparing a number of systems of same or different orders using pseudo-signature vectors of the competing systems, which is a distribution-free measure. The most interesting point in the results obtained here is that if the component reliabilities are not known or the reliability functions are difficult to obtain, we can still compare the system reliabilities making use of the stochastic ordering of their pseudo-signature vectors. This way, using the ordering of the pseudo-signatures, finally, we can arrange the systems in order of magnitude of their reliabilities.

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References

[1]. Barlow, R.E. and Proschan, F.(1981). *Statistical Theory of Reliability and Life Testing: Probability Models*. Silver Spring, MD.
[2]. Kochar, S., Mukerjee, H. and Samaniego, F. J.(1999). The "Signature" of a Coherent System and its Application to Comparisons among Systems. *Naval Research Logistics*, 46, 507-523.
[3]. Roychowdhury, S. and Bhattacharya, D.(2009). A Note on Comparison of Performances of Systems of Different Order. *Journal of Applied Statistical Science*, 16(3), 293-301.
[4]. Shanthikumar, J. G.(1987). On Stochastic Comparison of Random Vectors. *Journal of Applied Probability*, 24(1), 123-136.