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BAYESIAN ESTIMATION OF THE PARAMATER OF RAYLEIGH DISTRIBUTION UNDER THE EXTENDED JEFFREY'S PRIOR

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***Abstract**: In this paper, Bayes estimators of Rayleigh parameter and its associated risk based on extended Jeffrey's prior under the assumptions of both symmetric loss function (squared error loss) and asymmetric (precautionary and general entropy) loss function have been derived. We also derive the highest posterior density (HPD) and equal-tail prediction intervals for the parameter as well as the HPD prediction intervals for future observation. Monte Carlo simulations are performed to compare the performances of the Bayes estimates under different situations. Finally, an illustrative example is presented to assess how the Rayleigh distribution fits a real data set.*

***Keywords**: Bayes estimator, predictive distribution, predictive intervals, prior distribution.*

1. Introduction

The Rayleigh distribution was originally introduced by Lord Rayleigh [24] in the field of acoustics. Since then, many researchers have used this distribution in different field of science and technology. The Rayleigh distribution is frequently used to model wave heights in oceanography, and in communication engineering. Also, it has a wide application in lifetime data analysis especially in reliability theory and survival analysis. An important characteristic of the Rayleigh distribution is that its hazard rate is a linearly increasing function of time at constant rate which makes it a suitable model for the lifetime of components/items that age rapidly with time. Thus, as time increases, the reliability function of Rayleigh distribution decreases at a much higher rate than the exponential reliability function does.

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In this paper we consider one parameter Rayleigh distribution with the following probability density function (pdf):

$$f(t|\sigma) = \frac{t}{\sigma^2} exp\left(-\frac{t^2}{2\sigma^2}\right); \quad t > 0, \ \sigma > 0, \tag{1}$$

and corresponding cumulative distribution function:

$$F(t|\sigma) = 1 - exp\left(-\frac{t^2}{2\sigma^2}\right); \quad t > 0, \ \sigma > 0, \tag{2}$$

where σ is the scale parameter of the distribution. A great deal of research has been done on estimating the parameter of Rayleigh distribution using both classical and Bayesian techniques, and a very good summary of this work can be found in Sinha and Howlader [26], Ariyawansa and Templeton [5], Howlader [18], Howlader and Hossian [19], Lalitha and Mishra [20], Abd Elfattah et al. [1], Hendi et. al [17], Dey and Das [13] and Dey [12]. Statistical prediction was the earliest and most prevalent form of statistical inference. Prediction has its uses in variety of disciplines such as medicine, engineering and business. For more details on the history of statistical prediction analysis and examples, see Aitchison [2], Dunsmore [14], Aitchison and Dunsmore [3], Bain [6], Chhikara and Guttman [9], Geisser [15]. In this paper we consider the estimation of the posterior predictive density of a future observation based on the current data and construct predictive intervals of a future observation.

The rest of the paper is organized as follows. In section 2, we discuss prior and loss functions used in our Bayesian estimation. In section 3, we obtain the Bayes' estimators of σ and risk functions under symmetric and asymmetric loss functions. In section 4, Classical and Bayesian prediction intervals are obtained. In Section 5, we obtain the HPD interval for the Rayleigh parameter. Bayes predictive estimator and HPD prediction interval for a future observation are derived in Section 6. A simulation study is performed in Section 7. A real life data set is provided in Section 8 for the evaluation of Bayes estimates, classical and Bayesian prediction intervals and HPD prediction intervals for future observation and finally we conclude the paper in Section 9.

2. Prior and Loss functions

An important requisite in Bayesian estimation is the appropriate choice of prior(s) for the parameters. However, Bayesian analysts have pointed out that there is no clear cut way from which one can conclude that one prior is better than the other. Very often, priors are chosen according to ones subjective knowledge and beliefs. However, if one has adequate information about the parameter(s) one should use informative prior(s), otherwise it is preferable to use non-informative prior(s). In this paper we consider the extended Jeffrey's prior proposed by Al-Kutubi [4] as:

$$\pi(\sigma) \propto [I(\sigma)]^{c_1}, \quad c_1 \in R^+,$$

where $I(\sigma) = -nE\left(\frac{\partial^2 \ln f(t/\sigma)}{\partial\sigma^2}\right)$ is the Fisher's information matrix. For the model (1), $I(\sigma) = \frac{n}{\sigma^2}$ and hence:

$$\pi(\sigma) = k.\left[\frac{n}{\sigma^2}\right]^{c_1}, \tag{3}$$

where k is a constant. With the above prior, we use three different loss functions for the model (1): first is the squared error loss function which is symmetric, second is the precautionary loss which is a simple asymmetric function and third one is the general entropy loss function which is asymmetric in general and to some extent complex in nature.
It is well known that choice of loss function is an integral part of Bayesian inference. As there is no specific analytical procedure that allows us to identify the appropriate loss function to be used, most of the works on point estimation and point prediction assume the underlying loss function to be squared error which is symmetric in nature. However, in-discriminate use of SELF is not appropriate particularly in these cases, where the losses are not symmetric. Thus in order to make the statistical inferences more practical and applicable, we often needs to choose an asymmetric loss function. A number of asymmetric loss functions have been shown to be functional, see Varian [28], Zellner [29]. Moorhead and Wu [22], Spiring and Yeung [27], Chandra [8], etc. In the present work, we consider symmetric as well as asymmetric loss functions for better comprehension of Bayesian analysis.

a) The first is the common squared error loss function given by:

$$L_1(\hat{\sigma}, \sigma) = (\hat{\sigma} - \sigma)^2, \tag{4}$$

which is symmetric, and σ and $\hat{\sigma}$ represent the true and estimated values of the parameter. This loss function is frequently used because of its analytical tractability in Bayesian analysis.

b) The second is the precautionary loss function given by:

$$L_2(\hat{\sigma}, \sigma) = \frac{(\hat{\sigma} - \sigma)^2}{\hat{\sigma}}, \tag{5}$$

which is an asymmetric loss function, for details, see Norstrom [23]. This loss function is interesting in the sense that a slight modification of squared error loss introduces asymmetry.

c) The last loss function is the general entropy loss function which was proposed by Calabria and Pulcini [7] which is of the form

$$L_3(\hat{\sigma}, \sigma) = w\left[\left(\frac{\hat{\sigma}}{\sigma}\right)^p - p\ln\left(\frac{\hat{\sigma}}{\sigma}\right) - 1\right]; \quad w > 0,\ p \neq 0, \tag{6}$$

whose minimum occurs at $\hat{\sigma} = \sigma$. Without loss of generality, we assume $w = 1$. This loss is a generalization of the entropy loss used by several authors [see, for example, Dey et al.[10] and Dey and Liu [11]], where the value of the shape parameter p was taken as 1. If we replace $\hat{\sigma} - \sigma$ in place of $\ln(\frac{\hat{\sigma}}{\sigma})$ i.e. $\ln \hat{\sigma} - \ln \sigma$, we get the linear exponential (LINEX) loss function, $w\left[e^{p(\hat{\sigma}-\sigma)} - p(\hat{\sigma} - \sigma) - 1\right]$ which is proposed by Zellner [29].

3. Bayes Estimation of σ

Consider a group of n components put on test, their lifetimes $\underline{t} = (t_1, t_2, \ldots, t_n)$ are assumed to follow a Rayleigh distribution given in (1). The likelihood function of σ based on t is given by:

$$L(\underline{t}|\sigma) = \frac{1}{\sigma^{2n}} exp\left[-\frac{\sum_{i=1}^{n} t_i^2}{2\sigma^2}\right] \prod_{i=1}^{n} t_i = \frac{1}{\sigma^{2n}} exp\left[-\frac{s^2}{2\sigma^2}\right] \prod_{i=1}^{n} t_i, \tag{7}$$

where $s^2 = \sum_{i=1}^{n} t_i^2$. The conditional probability density function of σ given the data $\underline{t} = (t_1, t_2, \ldots, t_n)$ is given by:

$$\pi(\sigma|\underline{t}) = \frac{2\left(\frac{s^2}{2}\right)^{n+c_1-\frac{1}{2}}}{\Gamma\left(n + c_1 - \frac{1}{2}\right)(\sigma^2)^{n+c_1}} exp\left[-\frac{s^2}{2\sigma^2}\right]; \quad \sigma > 0. \tag{8}$$

Note that σ^2 follows an inverted gamma distribution, denoted as InGa(α, β), which has the following form of the pdf:

$$f(u) = \frac{\beta^\alpha}{\Gamma(\alpha)} u^{-(\alpha+1)} exp\left[-\frac{\beta}{u}\right]; \quad u > 0, \ \alpha, \beta > 0.$$

In our set up $\alpha = n + c_1 - \frac{1}{2}$ and $\beta = \frac{s^2}{2}$. By using squared error loss function (4), the risk function is:

$$R_s(\hat{\sigma}_s) = \int_0^\infty L_1(\hat{\sigma}_s, \sigma)\pi(\sigma|\underline{t})d\sigma$$
$$= \hat{\sigma}_s^2 - 2\hat{\sigma}_s \frac{\Gamma(n + c_1 - 1)}{\Gamma\left(n + c_1 - \frac{1}{2}\right)} \sqrt{\frac{s^2}{2}} + \frac{\Gamma\left(n + c_1 - \frac{3}{2}\right)}{\Gamma\left(n + c_1 - \frac{1}{2}\right)} \frac{s^2}{2}. \tag{9}$$

Bayes' estimator $\hat{\sigma}_s$ (subscript s stands for estimator under squared error loss function) is the solution of the equation, $\frac{\partial R(\hat{\sigma})}{\partial \hat{\sigma}}$ which implies,

$$\hat{\sigma}_s = \frac{\Gamma(n+c_1-1)}{\Gamma\left(n+c_1-\frac{1}{2}\right)}\sqrt{\frac{s^2}{2}}. \tag{10}$$

If $c_1 = \frac{1}{2}$, we get, the Jeffrey's prior and the corresponding Bayes estimator is:

$$\hat{\sigma}_s = \frac{\Gamma\left(n-\frac{1}{2}\right)}{\Gamma(n)}\sqrt{\frac{s^2}{2}}.$$

If $c_1 = \frac{3}{2}$, we get, the Hartigan prior [Hartigan [16]] and the corresponding Bayes estimator becomes:

$$\hat{\sigma}_s = \frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)}\sqrt{\frac{s^2}{2}}.$$

The Bayes estimator under a precautionary loss function (5) is denoted by $\hat{\sigma}_p$, and is given by the following equation:

$$\hat{\sigma}_p = [E(\sigma^2)]^{\frac{1}{2}}$$

and the corresponding Bayes estimator comes out to be:

$$\hat{\sigma}_s = \frac{\Gamma(n+c_1-1)}{\Gamma\left(n+c_1-\frac{1}{2}\right)}\sqrt{\frac{s^2}{2}}. \tag{11}$$

The risk function under precautionary loss function is given by:

$$R_p(\hat{\sigma}_p) = \hat{\sigma}_p - 2\frac{\Gamma(n+c_1-1)}{\Gamma\left(n+c_1-\frac{1}{2}\right)}\sqrt{\frac{s^2}{2}} + \frac{1}{\hat{\sigma}_p}\frac{\Gamma\left(n+c_1-\frac{3}{2}\right)}{\Gamma\left(n+c_1-\frac{1}{2}\right)}\frac{s^2}{2}. \tag{12}$$

The Bayes estimator under a general entropy loss function (6) is denoted by $\hat{\sigma}_g$, and is given by the following equation:

$$\hat{\sigma}_g = [E(\sigma^{-p})]^{-\frac{1}{p}}$$

and the corresponding Bayes estimator comes out to be:

$$\hat{\sigma}_g = \left[\frac{\Gamma\left(n + c_1 - \frac{1}{2}\right)}{\Gamma\left(n + c_1 + \frac{p}{2} - \frac{1}{2}\right)}\right]^{\frac{1}{p}} \sqrt{\frac{s^2}{2}}. \tag{13}$$

The risk function under general entropy loss function is given by:

$$R_g(\hat{\sigma}_g) = w\left[\frac{\Gamma(n + c_1 - 1)}{\Gamma\left(n + c_1 - \frac{1}{2}\right)}\left(\frac{s^2}{2\hat{\sigma}_g^2}\right)^{-\frac{p}{2}} + \frac{p}{2}\ln\frac{s^2}{2\hat{\sigma}_g^2} - \frac{p}{2}\phi\left(n + c_1 - \frac{1}{2}\right) - 1\right], \tag{14}$$

where $\phi(.)$ is the digamma function.

4. Highest Posterior Density Intervals for σ

In this section our objective is to provide a highest posterior density (HPD) interval for the unknown parameter σ of the model (1). Since the posterior density (8) is unimodal, the $100(1-\alpha)\%$ HPD interval $[H_1, H_2]$ for σ must satisfy:

$$\int_{H_1}^{H_2} \pi(\sigma|\underline{t})d\sigma = 1 - \alpha \tag{15}$$

and

$$\pi(H_1|\underline{t}) = \pi(H_2|\underline{t}) \tag{16}$$

simultaneously. After some algebra, the equations (15) and (16) take the following form:

$$\int_{\frac{s^2}{2H_2^2}}^{\frac{s^2}{2H_1^2}} \frac{1}{\Gamma\left(n + c_1 - \frac{1}{2}\right)} z^{n+c_1-\frac{3}{2}} e^{-z} dz = 1 - \alpha$$

$$\text{i.e. } \Gamma\left(n + c_1 - \frac{1}{2}, \frac{s^2}{2H_1^2}\right) - \Gamma\left(n + c_1 - \frac{1}{2}, \frac{s^2}{2H_2^2}\right) = 1 - \alpha \tag{17}$$

and

$$\left[\frac{H_2}{H_1}\right]^{2n+2c_1} = e^{\frac{s^2}{2H_1^2}-\frac{s^2}{2H_2^2}} \tag{18}$$

The HPD interval $[H_1, H_2]$ is the simultaneous solution of (17) and (18).

5. Predictive Distribution

In this section our objective is to obtain the posterior predictive density of future observation, based on current observations. We also aim to attain equal-tail Bayesian prediction interval for the future observation and then compare this interval with frequentist predictive interval. The posterior predictive distribution for $y = t_{n+1}$ given $\underline{t} = (t_1, t_2, \ldots, t_n)$ under (8) is defined by:

$$\xi(y|\underline{t}) = \int_0^\infty \pi(\sigma|\underline{t}) f(y|\sigma) d\sigma$$

$$= 2\left(n + c_1 - \frac{1}{2}\right) \frac{\frac{y}{s^2}}{\left(1+\frac{y^2}{s^2}\right)^{\left(n+c_1+\frac{1}{2}\right)}}. \tag{19}$$

A $100(1-\alpha)\%$ equal- tail prediction interval $[y_1, y_2]$ is the solution of:

$$\int_0^{y_1} \xi(y|\underline{t}) dy = \int_{y_2}^\infty \xi(y|\underline{t}) dy = \frac{\alpha}{2}.$$

Using (19), we get (after simplification):

$$= 2\left(n + c_1 - \frac{1}{2}\right) \frac{\frac{y}{s^2}}{\left(1+\frac{y^2}{s^2}\right)^{\left(n+c_1+\frac{1}{2}\right)}}. \tag{20}$$

and

$$y_2 = \left[\left(\frac{\alpha}{2}\right)^{-\frac{1}{n+c_1-\frac{1}{2}}} - 1\right]^{\frac{1}{2}} s. \tag{21}$$

Therefore,

$$I_{BJ} = y_2 - y_1 = \left\{\left[\left(\frac{\alpha}{2}\right)^{-\frac{1}{n+c_1-\frac{1}{2}}} - 1\right]^{\frac{1}{2}} - \left[\left(1-\frac{\alpha}{2}\right)^{-\frac{1}{n+c_1-\frac{1}{2}}} - 1\right]^{\frac{1}{2}}\right\} s,$$

where I_{BJ} refers to the length of predictive interval with respect to the extended Jeffrey's prior. For deriving classical intervals, we follow the approach of Sinha [25]. Since, the ratio $\frac{y^2}{s^2}$ is distributed as a beta-variate of the second kind with parameters 1 and n, the pdf of $z = \frac{y^2}{s^2}$ has the form:

$$h(z) = \frac{1}{B(1,\ n)(1+z)^{n+1}}; \quad z > 0.$$

Solving for $(z_1,\ z_2)$ in

$$\int_0^{z_1} h(z)dz = \int_{z_2}^{\infty} h(z)dz = \frac{\alpha}{2},$$

we have the length of classical interval (CI) for y [see Dey and Das [13]]

$$I_C = \left\{\left[\left(\frac{\alpha}{2}\right)^{-\frac{1}{n}} - 1\right]^{\frac{1}{2}} - \left[\left(1-\frac{\alpha}{2}\right)^{-\frac{1}{n}} - 1\right]^{\frac{1}{2}}\right\} s.$$

It is to be noted that if we take $c_1 = 0.5$ in (20) and (21), we get classical $100(1-\alpha)\%$ equal-tail prediction interval and hence get the length of that interval as I_C.

6. Bayes Predictive Estimator and HPD Prediction Interval for a Future Observation

The Bayes predictive estimator of y under a squared error loss function is:

$$y_1^* = E(y|\underline{t}) = \int_0^{\infty} y\,\xi(y|\underline{t})dy = \frac{\Gamma\left(\frac{3}{2}\right)\Gamma(n+c_1-1)}{\Gamma\left(n+c_1-\frac{1}{2}\right)} s. \tag{22}$$

The Bayes predictive estimator of y under the precautionary loss function (5) is given by

$$y_2^* = \left[\frac{\Gamma\left(n+c_1-\frac{3}{2}\right)}{\Gamma\left(n+c_1-\frac{1}{2}\right)}\right]^{\frac{1}{2}} s. \tag{23}$$

The Bayes predictive estimator of y under the general entropy loss function (6) is given by:

$$y_3^* = [J_g]^{-\frac{1}{p}}$$

$$= \left[\frac{\Gamma\left(1-\frac{p}{2}\right)\Gamma\left(n+c_1+\frac{p-1}{2}\right)}{\Gamma\left(n+c_1-\frac{1}{2}\right)}\right]^{-\frac{1}{p}} s, \quad \text{for } p < 2. \tag{24}$$

where:

$$J_g = \int_0^\infty y^{-p}\xi(y\,|\,\underline{t})dy$$

For the unimodal predictive density (19), the HPD-predictive interval $[h_1, h_2]$ with probability $1-\alpha$ for y is the simultaneous solution of the following:

$$P(h_1 < y < h_2) = 1-\alpha$$

$$\text{i. e. } \left[\frac{s^2}{s^2+h_1^2}\right]^{n+c_1-\frac{1}{2}} - \left[\frac{s^2}{s^2+h_2^2}\right]^{n+c_1-\frac{1}{2}} = 1-\alpha \tag{25}$$

and

$$\xi(h_1|\underline{t}) = \xi(h_2|\underline{t})$$

$$\text{i. e. } \left[\frac{s^2+h_2^2}{s^2+h_1^2}\right]^{n+c_1+\frac{1}{2}} = \frac{h_2}{h_1} \tag{26}$$

7. Simulation Study and Discussion

In this section, we conduct a simulation experiment in order to assess the performances of Bayes estimators of σ using the prior (2) under three different loss functions. The behavior of the loss functions is evaluated on the basis of risk estimates. The results of the simulation study are summarized in the Tables 1-3. We simulate samples from (1) with the true value of $\sigma = 1$, using four different sample sizes ($n = 10,20,30,100$). All results are based on 10000 repetitions. In the Tables, the estimators for the parameter and the risk, is averaged over the total number of repetitions. Monte Carlo standard deviation of each estimate is presented within parenthesis.

Table 1. Bayes estimate ($\hat{\sigma}_s$) and the corresponding risk estimate [$R_s(\hat{\sigma}_s)$] of the parameter σ under squared error loss.

	$n = 10$		$n = 20$		$n = 30$		$n = 100$	
c_1	$\hat{\sigma}_s$	$R_s(\hat{\sigma}_s)$	$\hat{\sigma}_s$	$R_s(\hat{\sigma}_s)$	$\hat{\sigma}_s$	$R_s(\hat{\sigma}_s)$	$\hat{\sigma}_s$	$R_s(\hat{\sigma}_s)$
0.5	1.027	0.030	1.012	0.014	1.009	0.009	1.002	0.003
1.0	1.001	0.027	1.001	0.013	1.000	0.009	0.999	0.003
1.5	0.974	0.025	0.988	0.012	0.991	0.008	0.997	0.002
2.0	0.949	0.022	0.975	0.012	0.985	0.008	0.995	0.002
5.0	0.846	0.014	0.914	0.009	0.940	0.007	0.980	0.002

Table 2. Bayes estimate ($\hat{\sigma}_p$) and the corresponding risk estimate [$R_p(\hat{\sigma}_p)$] of the parameter σ under precautionary loss.

	$n = 10$		$n = 20$		$n = 30$		$n = 100$	
c_1	$\hat{\sigma}_p$	$R_p(\hat{\sigma}_p)$	$\hat{\sigma}_p$	$R_p(\hat{\sigma}_p)$	$\hat{\sigma}_p$	$R_p(\hat{\sigma}_p)$	$\hat{\sigma}_p$	$R_p(\hat{\sigma}_p)$
0.5	1.040	0.029	1.020	0.013	1.012	0.009	1.003	0.003
1.0	1.015	0.027	1.007	0.013	1.004	0.008	1.001	0.003
1.5	0.987	0.025	0.994	0.012	0.995	0.008	0.999	0.003
2.0	0.965	0.023	0.982	0.012	0.988	0.008	0.996	0.002
5.0	0.851	0.016	0.914	0.010	0.943	0.007	0.981	0.002

Table 3. Bayes estimate ($\hat{\sigma}_g$) and the corresponding risk estimate [$R_g(\hat{\sigma}_g)$] of the parameter σ under general loss.

		$n = 10$		$n = 20$		$n = 30$		$n = 100$	
c_1	p	$\hat{\sigma}_g$	$R_g(\hat{\sigma}_g)$	$\hat{\sigma}_g$	$R_g(\hat{\sigma}_g)$	$\hat{\sigma}_g$	$R_g(\hat{\sigma}_g)$	$\hat{\sigma}_g$	$R_g(\hat{\sigma}_g)$
0.5	0.5	1.004	0.003	1.004	0.002	1.002	0.001	1.001	0.001
	1.0	1.000	0.013	1.001	0.006	1.000	0.004	1.000	0.001
	1.5	0.998	0.029	0.996	0.014	0.999	0.009	0.998	0.003
1.0	0.5	0.982	0.003	0.989	0.002	0.992	0.001	0.998	0.001
	1.0	0.976	0.012	0.987	0.006	0.992	0.004	0.998	0.001
	1.5	0.969	0.027	0.984	0.014	0.990	0.009	0.997	0.002
1.5	0.5	0.961	0.003	0.977	0.001	0.985	0.001	0.995	0.001
	1.0	0.955	0.012	0.976	0.006	0.984	0.004	0.996	0.001
	1.5	0.948	0.026	0.971	0.014	0.981	0.009	0.994	0.003
2.0	0.5	0.936	0.003	0.966	0.001	0.976	0.001	0.993	0.001
	1.0	0.928	0.011	0.964	0.006	0.974	0.004	0.993	0.001
	1.5	0.928	0.025	0.963	0.013	0.975	0.009	0.992	0.003
5.0	0.5	0.831	0.002	0.906	0.001	0.934	0.001	0.979	0.001
	1.0	0.827	0.009	0.902	0.005	0.931	0.004	0.978	0.001
	1.5	0.824	0.020	0.899	0.012	0.930	0.008	0.977	0.003

Tables 1-3 show that Bayes estimator under the choice of $c_1 = 0.5$ (Jeffrey's prior) provides little overestimation of the true parameter value in small sample situation, whereas under the choice of $c_1 = 1.5$ (Hartigan's prior), it underestimates even if sample size is quite large. When

c_1 takes large value, it has a tendency of underestimation in small sample setting, but with large sample size, we don't observe much change. We observe a general trend is that for a fixed c_1, the estimates and corresponding risks decrease with the increase in n. Same is true for fixed n but with the increase in c_1. There is not much variation in behavior of choice of loss functions as far as parameter estimation is concerned. From Table 3, it is observed that except for small c_1 and p, it has tendency of underestimation. Here the estimates of the parameter decrease but the corresponding risks increase slightly with the increase of p. As expected, when sample size increases, risk of estimator decreases. However, the rate of decrease in the risk varies with c_1. It is also observed that if the sample size is large, the effect of c_1 on the risk of the Bayes estimator is negligible.

Table 4. HPD intervals for the parameter σ of the Rayleigh distribution.

		$c_1 = 0.5$		$c_1 = 1.0$		$c_1 = 1.5$		$c_1 = 2.0$		$c_1 = 5.0$	
n	α	H_1	H_2	H_1	H_2	H_1	H_2	H_1	H_2	H_1	H_2
10	0.01	0.711	1.729	0.698	1.654	0.688	1.594	0.676	1.532	0.622	1.275
	0.05	0.764	1.492	0.752	1.442	0.739	1.392	0.725	1.345	0.666	1.145
	0.10	0.711	1.729	0.698	1.654	0.688	1.594	0.676	1.532	0.622	1.275
20	0.01	0.777	1.417	0.769	1.391	0.762	1.369	0.756	1.348	0.717	1.230
	0.05	0.819	1.292	0.811	1.270	0.806	1.255	0.798	1.237	0.757	1.139
	0.10	0.838	1.227	0.832	1.211	0.826	1.197	0.819	1.182	0.778	1.096
30	0.01	0.810	1.314	0.804	1.299	0.800	1.287	0.795	1.274	0.767	1.202
	0.05	0.847	1.222	0.842	1.211	0.838	1.201	0.833	1.191	0.801	1.127
	0.10	0.864	1.175	0.860	1.167	0.856	1.158	0.849	1.146	0.821	1.092
100	0.01	0.886	1.149	0.884	1.145	0.882	1.142	0.880	1.140	0.870	1.121
	0.05	0.909	1.109	0.908	1.106	0.906	1.103	0.904	1.100	0.893	1.084
	0.10	0.921	1.087	0.918	1.083	0.917	1.082	0.916	1.080	0.905	1.065

In Table 4, the HPD interval estimates for σ have been reported. We have analyzed these intervals for three different coverage probabilities (99%, 95%, 90%). As expected, it is observed that as the sample size increases, the HPD interval becomes narrower so is the case when coverage probability decreases. There is little difference in the behavior of HPD intervals when comparing Jeffrey's prior (when $c_1 = 0.5$) over Hartigan's prior (when $c_1 = 1.5$). The intervals are more or less same for large samples.

Table 5 indicates that the Bayes predictive intervals (within parenthesis) are favorable than the classical predictive intervals as it offers much shorter intervals than the classical intervals. The predictive estimates for future observation and the corresponding HPD intervals have been reported in Table 6. Here y_1^*, y_2^* and y_3^* are derived by using squared error loss, precautionary loss and general entropy loss respectively. The difference between y_1^* and y_2^* are not so prominent but y_3^* significantly differs from these two. For all predictive estimates consistency property is verified with the increase of sample size and with the increase of c_1. HPD intervals provide fairly reasonable coverage for all the estimates from shortest length of interval point of view.

Table 5. Classical (Bayesian) predictive interval (y_1, y_2) for the future observation from the Rayleigh distribution.

		$c_1 = 0.5$		$c_1 = 1.0$		$c_1 = 1.5$		$c_1 = 2.0$		$c_1 = 5.0$	
n	α	y_1	y_2	y_1	y_2	y_1	y_2	y_1	y_2	y_1	y_2
10	0.01	0.451 (0.140)	16.846 (5.243)	0.438 (0.139)	16.231 (5.160)	0.427 (0.135)	15.742 (4.965)	0.417 (0.131)	15.285 (4.817)	0.371 (0.117)	13.262 (4.181)
	0.05	1.011 (0.318)	13.408 (4.221)	0.985 (0.311)	13.003 (4.119)	0.961 (0.300)	12.639 (3.949)	0.943 (0.301)	12.359 (3.948)	0.838 (0.267)	10.788 (3.436)
	0.10	1.435 (0.457)	11.832 (3.766)	1.400 (0.442)	11.497 (3.637)	1.364 (0.435)	11.166 (3.557)	1.333 (0.428)	10.881 (3.490)	1.191 (0.370)	9.585 (2.980)
20	0.01	0.632 (0.141)	21.969 (4.920)	0.626 (0.141)	21.729 (4.883)	0.619 (0.138)	21.479 (4.778)	0.611 (0.138)	21.184 (4.700)	0.574 (0.129)	19.725 (4.427)
	0.05	1.421 (0.315)	17.973 (3.978)	1.404 (0.314)	17.738 (3.960)	1.387 (0.309)	17.506 (3.893)	1.371 (0.307)	17.274 (3.870)	1.287 (0.287)	16.131 (3.594)
	0.10	2.027 (0.452)	16.076 (3.586)	1.994 (0.445)	15.803 (3.526)	1.974 (0.445)	15.633 (3.523)	1.954 (0.435)	15.461 (3.442)	1.830 (0.409)	14.420 (3.222)
30	0.01	0.775 (0.141)	26.364 (4.795)	0.770 (0.140)	26.174 (4.770)	0.763 (0.139)	25.921 (4.710)	0.758 (0.139)	25.685 (4.709)	0.722 (0.132)	24.421 (4.451)
	0.05	1.745 (0.320)	21.719 (3.981)	1.721 (0.317)	21.418 (3.945)	1.712 (0.313)	21.295 (3.896)	1.700 (0.311)	21.133 (3.860)	1.627 (0.294)	20.169 (3.640)
	0.10	2.484 (0.453)	19.462 (3.549)	2.468 (0.454)	19.324 (3.556)	2.447 (0.448)	19.153 (3.504)	2.421 (0.449)	18.942 (3.510)	2.312 (0.424)	18.053 (3.311)
100	0.01	1.414 (0.140)	46.576 (4.626)	1.414 (0.142)	46.568 (4.665)	1.409 (0.143)	46.406 (4.716)	1.406 (0.142)	46.300 (4.672)	1.386 (0.139)	45.631 (4.580)
	0.05	3.180 (0.319)	38.742 (3.884)	3.175 (0.318)	38.678 (3.871)	3.169 (0.314)	38.603 (3.825)	3.160 (0.319)	38.486 (3.887)	3.112 (0.311)	37.895 (3.786)
	0.10	4.536 (0.454)	34.920 (3.498)	4.513 (0.447)	34.747 (3.440)	4.513 (0.456)	34.744 (3.509)	4.500 (0.449)	34.640 (3.453)	4.432 (0.444)	34.108 (3.414)

Table 6. Estimated future observation from the Rayleigh distribution under three losses and HPD predictive interval for future observation.

n	c_1	y_1^*	y_2^*	y_3^*			α =0.01		α =0.05		α =0.10	
				p =0.5	p =1.0	p =1.5	H_1	H_2	H_1	H_2	H_1	H_2
10	0.5	1.291	1.478	0.947	0.798	0.598	0.022	3.552	0.092	2.713	0.160	2.321
	1.0	1.252	1.431	0.925	0.778	0.580	0.020	3.245	0.091	2.643	0.157	2.263
	1.5	1.220	1.394	0.902	0.761	0.566	0.021	3.351	0.089	2.576	0.155	2.207
	2.0	1.292	1.361	0.879	0.743	0.554	0.021	3.255	0.088	2.511	0.152	2.153
	5.0	1.054	1.201	0.782	0.660	0.493	0.019	2.744	0.081	2.211	0.139	1.900
20	0.5	1.269	1.442	0.944	0.798	0.597	0.024	3.249	0.101	2.609	0.173	2.277
	1.0	1.253	1.423	0.933	0.789	0.590	0.025	3.192	0.100	2.570	0.171	2.246
	1.5	1.237	1.404	0.923	0.777	0.582	0.025	3.180	0.099	2.543	0.170	2.221
	2.0	1.224	1.390	0.911	0.770	0.577	0.024	3.128	0.098	2.510	0.168	2.190
	5.0	1.143	1.297	0.851	0.720	0.539	0.023	2.919	0.093	2.343	0.159	2.048
30	0.5	1.263	1.432	0.944	0.799	0.598	0.026	3.195	0.104	2.572	0.178	2.263
	1.0	1.254	1.421	0.936	0.792	0.593	0.026	3.154	0.103	2.549	0.176	2.244
	1.5	1.244	1.409	0.926	0.784	0.589	0.026	3.122	0.103	2.528	0.175	2.223
	2.0	1.235	1.400	0.920	0.780	0.585	0.026	3.109	0.102	2.509	0.174	2.205
	5.0	1.176	1.332	0.881	0.744	0.557	0.025	2.954	0.098	2.390	0.167	2.103
100	0.5	1.257	1.420	0.942	0.797	0.599	0.028	3.085	0.108	2.520	0.184	2.239
	1.0	1.253	1.416	0.940	0.796	0.597	0.028	3.079	0.108	2.517	0.184	2.236
	1.5	1.251	1.413	0.938	0.794	0.596	0.028	3.075	0.108	2.510	0.183	2.231
	2.0	1.247	1.409	0.936	0.792	0.594	0.028	3.061	0.108	2.503	0.183	2.224
	5.0	1.228	1.387	0.922	0.781	0.585	0.028	3.016	0.106	2.465	0.180	2.192

8. Data Analysis

For illustrative purposes, we consider the following real data set which arose in tests on endurance of deep grove ball bearings [Lawless [21], p.228]. The data are the number of hundreds of million revolutions before failure for each of the 23 ball bearings in the life test:

0.1788, 0.2892, 0.33, 0.4152, 0.4212, 0.456, 0.488, 0.5184, 0.5196, 0.5412, 0.5556, 0.678, 0.6864, 0.6864, 0.6888, 0.8412, 0.9312, 0.9864, 1.0512, 1.0584, 1.2792, 1.2804, 1.734.

To study the goodness of fit of the Rayleigh model, we compute the χ^2 statistic and it is 0.1225 with the corresponding p-value is 0.7264. Therefore, the high p-value clearly suggests that one parameter Rayleigh model can be used to analyze this data set. Here, our goal is to estimate the Rayleigh parameter for this data set under three different losses. At the same time, we are interested to study the HPD intervals for the parameter. We also find the future observation based on the given data and estimated predictive distribution is shown in Figure 1, and look into the behavior of HPD intervals for the predicted observation. Tables 7-10 summarize the findings. From Table 7, it is observed that the estimates of σ are very closer and the corresponding risks are fairly small except for general entropy loss at $p = 1.5$. HPD intervals are fair enough from coverage point of view (Table 8). Though the estimated future observations (Table 9) are not so encouraging, HPD intervals (second intervals in each row) for future observation are reasonably good (Table 10).

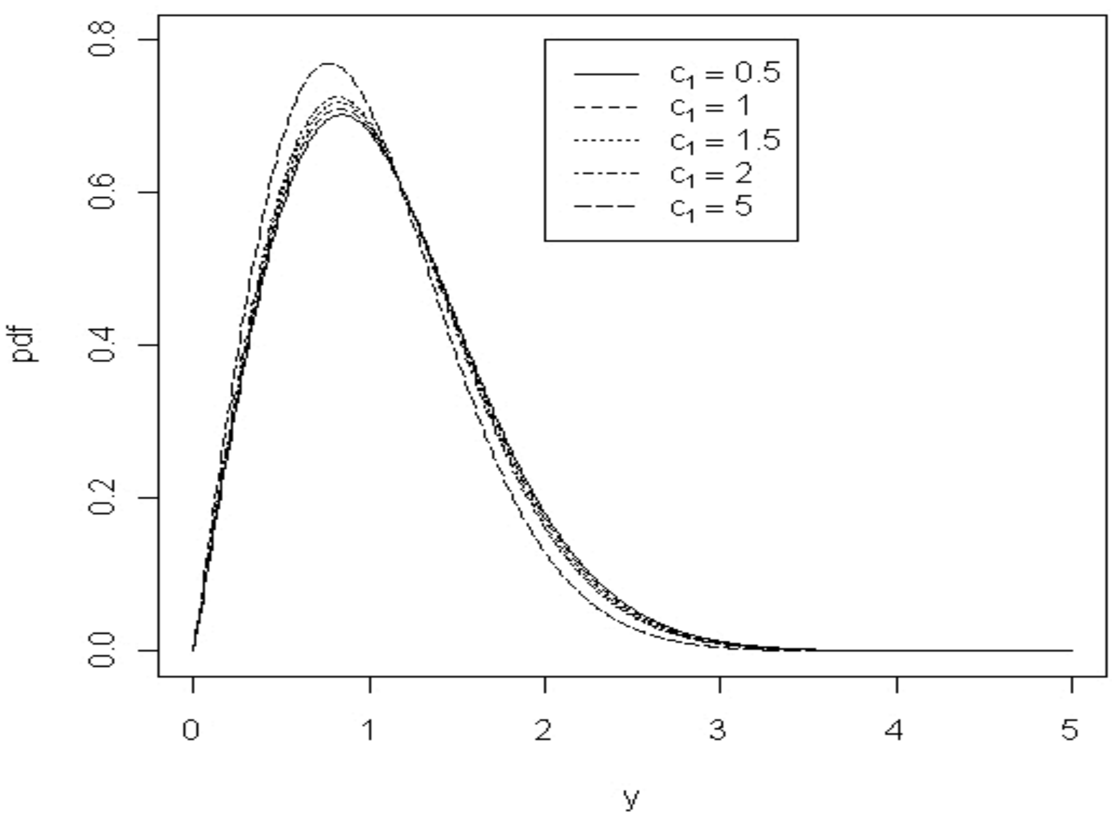

Figure 1. Estimate of the predictive distribution for the given data set.

Table 7. Bayes estimates of the parameter and the corresponding risk estimates under three losses for the data set.

c_1	$\hat{\sigma}_s$	$R_s(\hat{\sigma}_s)$	$\hat{\sigma}_p$	$R_p(\hat{\sigma}_p)$	p =0.5		p =1.0		p =1.5	
					$\hat{\sigma}_g$	$R_g(\hat{\sigma}_g)$	$\hat{\sigma}_g$	$R_g(\hat{\sigma}_g)$	$\hat{\sigma}_g$	$R_g(\hat{\sigma}_g)$
0.5	0.582	0.0039	0.586	0.0066	0.578	0.00138	0.576	0.0055	0.574	0.0124
1.0	0.576	0.0037	0.579	0.0064	0.571	0.00135	0.570	0.0054	0.568	0.0120
1.5	0.570	0.0035	0.573	0.0062	0.566	0.00132	0.564	0.0053	0.562	0.0118
2.0	0.564	0.0034	0.567	0.0060	0.559	0.00130	0.558	0.0052	0.556	0.0116
5.0	0.531	0.0027	0.534	0.0050	0.527	0.00115	0.526	0.0046	0.525	0.0103

Table 8. HPD intervals of the parameter for the data set.

c_1	(H_1, H_2)		
	α =0.01	α =0.05	α =0.10
0.5	(0.455, 0.794)	(0.481, 0.734)	(0.496, 0.706)
1.0	(0.451, 0.782)	(0.477, 0.724)	(0.491, 0.697)
1.5	(0.448, 0.770)	(0.473, 0.714)	(0.487, 0.688)
2.0	(0.444, 0.759)	(0.469, 0.705)	(0.482, 0.679)
5.0	(0.423, 0.702)	(0.446, 0.655)	(0.458, 0.632)

Table 9. Estimated future observation for the data set under three losses.

c_1	y_1^*	y_2^*	y_3^*		
			p =0.5	p =1.0	p =1.5
0.5	0.730	0.828	0.544	0.460	0.344
1.0	0.722	0.819	0.538	0.455	0.340
1.5	0.714	0.810	0.532	0.450	0.337
2.0	0.706	0.801	0.527	0.445	0.333
5.0	0.666	0.755	0.497	0.420	0.315

Table 10. Classical, Bayesian predictive and HPD predictive intervals for future observation of the data set.

α	CI	$c_1 = 0.5$	$c_1 = 1.0$	$c_1 = 1.5$	$c_1 = 2.0$	$c_1 = 5.0$
0.01	(0.223, 7.682)	(0.223, 7.682)	(0.220, 7.589)	(0.218, 7.501)	(0.216, 7.416)	(0.204, 6.957)
		(0.081, 2.838)	(0.080, 2.717)	(0.079, 2.616)	(0.078, 2.529)	(0.074, 2.498)
0.05	(0.501, 6.295)	(0.501, 6.295)	(0.496, 6.222)	(0.490, 6.152)	(0.485, 6.084)	(0.458, 5.718)
		(0.183, 2.527)	(0.181, 2.482)	(0.179, 2.423)	(0.177, 2.356)	(0.167, 2.193)
0.10	(0.713, 5.629)	(0.713, 5.629)	(0.706, 5.565)	(0.698, 5.503)	(0.691, 5.443)	(0.652, 5.121)
		(0.264, 2.498)	(0.260, 2.491)	(0.257, 2.369)	(0.255, 2.339)	(0.240, 2.177)

9. Concluding Remark

In this article, we have primarily studied the Bayes estimator of the parameter of the Rayleigh distribution under the extended Jeffrey's prior assuming three different loss functions. The estimates are reasonably good under the simulation study and the real data analysis. The extended Jeffrey's prior gives the opportunity of covering wide spectrum of priors to get Bayes estimates of the parameter - particular cases of which are Jeffrey's prior and Hartigan's prior. The HPD intervals allow quite reasonable coverage for the estimates of the Rayleigh parameter. It is also noticed that the Bayes predictive intervals are favorable than the classical predictive intervals as it offers much shorter intervals than the classical intervals. The HPD intervals for future observation are also quite good. Real life data analysis echoes the same trend that has been observed in the simulation study.

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