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THE BIPOLAR MEAN IN SENSORY ANALYSIS

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Abstract: *The bipolar mean has recently been put forward with the aim of summarizing ordinal variables* [1]. *It is a synthetic distribution where the total size n is concentrated on one of the k categories of the variable or, at most, on two consecutive categories. This measure is derived according to the usual statistical dominance criterion that is based on retro-cumulative frequencies. Further improvements to the bipolar mean include extensions to discrete quantitative variables and a new variability measure, i.e. the "mean deviation about the bipolar mean". The bipolar mean can also be applied to ordinal variables whose categories are expressed as scores on a numerical scale. Hence, this new way of summarizing such variables can be useful in sensory analysis, where it is often necessary to compare frequency distributions representing the evaluation of certain characteristics made by judges or tasters about different products. The assessment is usually based on simple synthetic measures such as the arithmetic mean or the median. These indexes however, can result in contradictory answers.*
In this work, we present the normalization of the mean deviation of the bipolar mean. Moreover, some empirical evidence in sensory analysis is given with the purpose of showing how the bipolar mean and the relative mean deviation can sometimes overcome these problems of comparison.

Keywords: *Bipolar mean, statistical dominance, retro-cumulative frequencies, mean deviation about the bipolar mean, maximum value of the mean deviation about the bipolar mean, sensory analysis.*

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1. Introduction

In sensory analysis, it is often necessary to compare rating scale scores given by a panel of n assessors (*judges*, *tasters*, *consumers*) for certain product characteristics (*descriptors*).
Suppose, for example, that seven judges give scores from 1 (*very bad*) to 5 (*very good*) for two descriptors (A and B) of three "*grappe*" (G_1, G_2, G_3) [2]: the aim is to form a ranking of the three *grappe* for each descriptor. Usually, the comparison is made using simple indexes such as the median[1] (*Me* henceforth) or the arithmetic mean (μ henceforth).
In Table 1 we show the rankings obtained with these two indexes. It is clear that when we use the *Me*, the three *grappe* are equivalent in the case of descriptor A while G_{3B} is preferred in the case of descriptor B. The conclusions reached are reversed when μ is used: the three *grappe* are equivalent in the case of descriptor B, while G_{3A} is preferred in the case of descriptor A.

Table 1. Score distributions, arithmetic means and medians (k = 5; n = 7)

Scores	Descriptor A			Descriptor B		
	G_{1A}	G_{2A}	G_{3A}	G_{1B}	G_{2B}	G_{3B}
1	3	2	2	1	0	1
2	3	3	4	3	1	2
3	0	2	0	0	5	0
4	1	0	0	1	1	4
5	0	0	1	2	0	0
Me	**2**	**2**	**2**	**2**	**3**	**4**
μ	**1,857**	**2**	**2,143**	**3**	**3**	**3**

This type of problem can be resolved by using the bipolar mean. W. Maffenini and Michele Zenga [5] proposed the bipolar mean as a synthesis for ordinal qualitative characters. Later Maffenini and Mariangela Zenga [6] extended the bipolar mean to discrete variables and they put forward a new variability measure: the "mean deviation about the bipolar mean" that can also be computed for ordinal qualitative characters whose categories are expressed on a ranking scale as shown in our example.
The mean deviation about the bipolar mean is an absolute index of variability and as for all these types of indexes it is useful to set its maximum value. To do this, we define its maximum variability distribution [3], then we derive the maximum bipolar mean and the maximum value of the index.
This article is structured as follows: in Section 2 there are some methodological details about the bipolar mean, the mean deviation about the bipolar mean and its maximum. In Section 3 we introduce an example to show the usefulness of this index in the framework of sensory analysis and Section 4 concludes the paper.

[1] The *Me* is often preferred since it is less sensitive to outlying scores, i.e. sporadic cases which are distant from the scores given by the majority of the judges.

2. The Bipolar Mean

The bipolar mean is a distribution where the total size n is concentrated on a single category of an ordered distribution (or a single value of a discrete variable) or, at most, on two adjacent categories (values), and it is coherent with the usual statistical dominance criterion based on retro-cumulative frequencies (order W).
Each empirical distribution has its own corresponding bipolar mean which synthesizes the distribution and satisfies the ordering requirement.
Let X be a discrete variable, taking the values 1, 2, …, s,…, k and let n_1, n_2,…, n_s,…, n_k ($n = \sum_{s=1}^{k} n_s$) be the corresponding frequencies.
Let $\mathcal{B}$ be the collection of all possible distributions that satisfy the constraints:

$$n_s \geq 0, \qquad s = 1, 2, \ldots, k; \qquad \sum_{s=1}^{k} n_s = n \,.$$

The number of distributions of $\mathcal{B}$ is $\binom{n+k-1}{n}$.

The distributions of $\mathcal{B}$ can be compared according to the principle of statistical dominance that is based on decreasing cumulative frequencies:

$$R_S = \sum_{j=s}^{k} n_j \qquad s = 1, 2, \ldots, k \,.$$

Let $(n_1, n_2, \ldots, n_k)$ and $(n_1', n_2', \ldots, n_k')$ be two distributions belonging to $\mathcal{B}$. The distribution $(n_1, n_2, \ldots, n_k)$ is greater than the distribution $(n_1', n_2', \ldots, n_k')$, i.e.:

$$(n_1', n_2', \ldots, n_k') \underset{W}{\prec} (n_1, n_2, \ldots, n_k)$$

If:

$$R_S' \leq R_S \qquad s = 1, 2, \ldots, k$$

with at least one strict inequality. Usually, it is not possible to order all the distributions belonging to $\mathcal{B}$ according to W. However, we can focus on a subset $\mathcal{B}$* which only contains distributions with the following characteristics:

i. n is concentrated on only one of the k values of X;
ii. n is concentrated on two adjacent values of X.

The number of distributions of $\mathcal{B}$* is nk–n+1.

A function *H*, coherent with *W*, is introduced with the aim of ordering all the distributions of $\mathcal{B}$ and then also the distributions not belonging to $\mathcal{B}$*, i.e.:

$$\left(n_1^{'},...,n_k^{'}\right) \underset{W}{\prec} \left(n_1,...,n_k\right) \;\; \Rightarrow \;\; H\left(n_1^{'},...,n_k^{'}\right) < H\left(n_1,...,n_k\right). \qquad (1)$$

The sum of the retro-cumulative frequencies:

$$G = G\left(n_1,...,n_k\right) = \sum_{s=1}^{k} R_s = \sum_{s=1}^{k}\sum_{j=s}^{k} n_j$$

is a function that satisfies the relation (1). It is easy to verify that, in correspondence with the *nk*–*n*+1 distributions belonging to $\mathcal{B}$*, sorted in ascending order, *G* takes values *n*, *n*+1, …, 2*n*–1, 2*n*, 2*n*+1,…, *kn*–1, *kn*.
The value *n* comes from the distribution $\left(n, 0, \ldots, 0\right)$, …, the value *kn* from $\left(0, \ldots, 0, n\right)$. Furthermore, the values *n*, 2*n*, 3*n*, ..., *kn* derive from those distributions for which the total size *n* is concentrated on one value of *X*, while the remaining values come from those distributions where *n* belongs to two adjacent values of *X*.
It is possible to demonstrate that, for all distributions belonging to $\mathcal{B}$, the function *G*:

(*i*) assumes integer values in the range [*n*, *kn*], i.e. those values that correspond to the distributions belonging to $\mathcal{B}$*;
(*ii*) may be expressed as:

$$G = \sum_{s=1}^{k} R_s = \sum_{s=1}^{k} s\, n_s \qquad (2)$$

It follows that, in correspondence with the distributions of $\mathcal{B}$ that do not belong to $\mathcal{B}$*, *G* takes values in the interval [*n*, *kn*]. Hence the collection $\mathcal{B}$ is shared among *nk*–*n*+1 subsets (i.e., the same number of distributions that belong to $\mathcal{B}$*). The function *G* assigns the value *g* to all the distributions belonging to the same subset which includes a single member of $\mathcal{B}$*. Therefore, these distributions are equivalent according to the order *W*.
It is possible then, to represent all distributions of the same subset of $\mathcal{B}$ with the unique distribution belonging to $\mathcal{B}$*, since it is the "ultimate synthesis" as it puts the total size *n* on a single value of *X* or, at most, on two adjacent values.
This distribution is the *Bipolar Mean* (*BM*).
To obtain the *BM* of a frequency distribution it is useful to consider the function:

$$\bar{S} = \frac{1}{n} G = \frac{1}{n} \sum_{s=1}^{k} R_s = \frac{1}{n} \sum_{s=1}^{k} s \cdot n_s = \sum_{s=1}^{k} s \cdot f_s \qquad (3)$$

where $f_s = n_s / n$.
From (3) it is evident that $\bar{S}$ is the weighted arithmetic mean of *X* and it assumes the values 1, 1+1/*n*,…, 2–1/*n*, 2, 2+1/*n*,…, *k*.

Obviously, the function $\bar{S}$ is coherent with the order *W* and all the properties for the $\mathcal{B}$ subsets also hold for this case. $\bar{S}$ takes the integer values 1, 2, …, *k* on those distributions where the total size *n* is concentrated on a single value of *X* (*BM* of type I) and the decimals on those distributions where *n* is divided between two adjacent values (*BM* of type II).
To obtain the *BM* we can proceed as follows:

(*i*) when $\bar{S}$ is integer (1, 2,..., *s*, …, *k*), the *BM* puts the total size *n* on the corresponding values 1, 2,..., *s*, …, *k*;

(*ii*) when $\bar{S}$ is a non-integer between *s* and *s*+1, *s*=1, …, (*k*-1), the *BM* puts on the value *s*+1 the frequency n_{s+1} that corresponds to the product of the fractional part of $\bar{S}$ and *n*, and on the value *s* the frequency $n_s = n - n_{s+1}$.

2.1 The Mean Deviation about the Bipolar Mean

Let η be the *BM* of *X*; let N_s be the *s*-th cumulative frequency of *X*; let $\tilde{N}_s$ be *s*-th cumulative frequency of η. The Mean Deviation about *BM* is given by:

$$S_\eta = \frac{1}{n}\sum_{s=1}^{k}\left|N_s - \tilde{N}_s\right|. \tag{4}$$

W. Maffenini and Mariangela Zenga (2006) [6] demonstrated that S_η and the Mean Deviation about the Arithmetic Mean S_μ are related as follows:

$$S_\eta \le S_\mu \tag{5}$$

where the equality holds *iff* $\mu = s$, $s = 1, 2, \dots, k$ namely in the case of *BM* of type I.

2.2 Maximum variability distribution and maximum Mean Deviation about the Bipolar Mean

To make a comparison, the Mean Deviation about the *BM* can be expressed as a relative measure. This can be done in two ways: (*i*) dividing the *BM* by an appropriate mean; (*ii*) dividing the *BM* by the value that it assumes in a maximum variability distribution.
In (*i*), it would seem appropriate to choose the arithmetic mean, taking into account its links with the function that determines the *BM*. In (*ii*), first it is necessary to define the maximum variability distribution that is different depending on whether the number of cases (*n*) is even or odd. Once this is derived, the corresponding *BM* is computed, taking into account that also the maximum value of *X* (*k*), (obviously the same when comparing distributions), may be even or odd.

Maximum variability distribution for discrete variables (definition)

To distinguish the cases even or odd, we suggest indicating the even number with *n* and the odd number with $m = n + 1$; the corresponding generic frequencies will be indicated as n_s and m_s.
Let *X* be a discrete variable that takes values 1, 2, ..., *s*, ..., *k* with frequencies whose sum may be even or odd.

Case 1) Odd number: $n=\sum_{s=1}^{k} n_s$. The maximum variability distribution of X is

(1) (n/2, 0,…, 0,…, n/2)

Case 2) Even number: $m=n+1=\sum_{s=1}^{k} m_s$. The two maximum variability distributions of X are:

(2a) (n/2+1, 0,…, 0,…, n/2);

(2b) (n/2, 0,…, 0,…, n/2+1).

Let S_η^* be the maximum of S_η, we determine this with the bipolar mean η* representing the value $\bar{s}*$ (i.e. the value taken by the function $\bar{s}$ in correspondence with the maximum variability distribution) or, identically, with the value μ^* (i.e. the value taken by μ in correspondence with this distribution [see (3)]). From the equation (4) of S_η we immediately derive the expression for S_η^*

$$S_\eta^* = \frac{1}{n}\sum_{s=1}^{k}\left|N_s^* - \tilde{N}_s^*\right| \qquad (6)$$

where N_s^* and $\tilde{N}_s^*$ are the cumulative frequencies obtained respectively, from the maximum variability distribution and its *BM* η*. As mentioned before, to find the maximum of S_η^* we have to consider that k can be even or odd. Only in the case «odd k, even n» the *BM* of the maximum variability distribution is of type I, while in all the other cases it is of type II.

Case A: «odd k, even n»
If n is even, it can be equally shared between the smallest value 1 and the largest k. Since the arithmetic mean $\mu^* = (k+1)/2$ is an integer, the *BM* is of type I and puts the total size n on the value (k+1)/2. The maximum mean deviation about the bipolar mean is:

$$S_\eta^* = \frac{k-1}{2}.$$

Case B: «odd k, odd n»
If n is odd there are two maximum variability distributions since n cannot be equally shared between the extreme values 1 and k. In fact, one frequency must be assigned to the smallest value 1 (case B1) or to the largest value k (case B2). In both cases, the maximum variability mean is not an integer and the bipolar mean is of type II. The maximum mean deviation about the bipolar mean is, for both cases B1) and B2):

$$S_\eta^* = \frac{k-1}{2}\frac{n-1}{n}.$$

The result is similar to that achieved in case *A* with even *n*, but now the term (*k*-1)/2 is multiplied by the corrective factor (*n*-1)/*n*.

Case C: **«**even *k*, even *n***»**
If both *n* and *k* are even, the arithmetic mean $\mu^* = (k+1)/2$ is not an integer and therefore the *BM* is of type II. The maximum mean deviation about the bipolar mean is:

$$S_\eta^* = \frac{k-2}{2}.$$

Case D: «even *k*, odd *n*»
If *k* is even and *n* is odd, the mean of the maximum variability distribution is not an integer: *n* cannot be equally shared between the extreme values. Consequently, one frequency must be assigned to the smallest value 1 or to the largest value *k*. In both cases the *BM* is of type II.
The maximum mean deviation about the bipolar mean is:

$$S_\eta^* = \frac{k-2}{2}\frac{n+1}{n}.$$

The result is similar to that achieved in case *C* with even *n*, but now the term (*k*-2)/2 is multiplied by the corrective factor (*n*+1)/*n*. The four cases are summarized in Table 2.

Table 2. Expressions of S_η^* given *n* (number of statistical unities) and *k* (maximum value of *X*)

	even *n*	**odd *n***
odd *k*	$\frac{k-1}{2}$	$\frac{k-1}{2}\frac{n-1}{n}$
even *k*	$\frac{k-2}{2}$	$\frac{k-2}{2}\frac{n+1}{n}$

3. Examples

To show how the bipolar mean can be used in sensory analysis, we consider the same example discussed in the introduction. We compute the *BM*, according to the explanation in Section 2, for the score distributions of seven judges (see Table 1) regarding two descriptors (*A* and *B*) relative to three *grappe* (G_1, G_2, G_3).
As an example, we show the calculus of the *BM* for G_{1A} and G_{2A}. In the case of G_{1A}, given that $\bar{s}$=1,857, we use the rule *ii*) which assigns the frequency $7 \times 0{,}875 = 6$ to score 2 and the frequency 7-6=1 to score 1. From Table 3, it is clear that the *BM* (1, 6, 0, 0, 0) concentrates 86% of the frequencies on score 2 and 14% on score 1 while the *MB* (0, 7, 0, 0, 0) concentrates 100% of the frequencies on score 2. We found that the *BMs* of the other distributions followed the same method.

Table 3. Score distributions and their bipolar means and synthetic indexes

Scores	Descriptor A			Descriptor B		
	$BM(G_{1A})$	$BM(G_{2A})$	$BM(G_{3A})$	$BM(G_{1B})$	$BM(G_{2B})$	$BM(G_{3B})$
1	1	0	0	0	0	0
2	6	7	6	0	0	0
3	0	0	1	7	7	7
4	0	0	0	0	0	0
5	0	0	0	0	0	0
$\bar{s}$ (μ)	**1,857**	**2**	**2,143**	**3**	**3**	**3**
S_η	**0,571**	**0,571**	**0,571**	**1,429**	**0,286**	**1,143**
S_η^*	$\frac{5-1}{2}\times\frac{6}{7}$ **=1,714**	**1,714**	**1,714**	**1,714**	**1,714**	**1,714**
$S_\eta / S_\eta^* \%$	**33,31%**	**33,31%**	**33,31%**	**83,37%**	**16,69%**	**66,69%**

The distributions G_{1A}, G_{2A} et G_{3A} belong to three different subsets of $\mathcal{B}$ and they are summarized by the *BM* of the corresponding subset. We can see that the order of the *BM* is the same as that given by μ. Note that since *BM* is a frequency distribution it can give us broader interpretations relative to μ which instead, is a single value.
The situation is quite different for the descriptor *B*, whose three distributions are summarized by the same *BM*. In fact, since $\bar{s}$=3 for G_{1B}, G_{2B} and G_{3B} and using the rule *i*) (that assigns all the frequency to the same score) we find a *BM* (0, 0, 7, 0, 0) that concentrates 100% of the frequencies on score 3. It is therefore not possible to establish an order for the three distributions that are equivalent according to the order *W*.
In this case, it is useful to consider the values of the mean deviation about the *BM* which are different for each one of the three distributions (Table 3). The index is 83,37% of its theoretical maximum for G_{1B} , 16,69% for G_{2B} and 66,69% for G_{3B}. Hence, it is possible to rank the judgements on the three *grappe* even when they have the same *BM*. This is possible because we take into account the variability of judgements given by the seven judges. We will draw the conclusion that the distribution G_{2B} is better than G_{3B} and the latter is better than G_{1B}.

3.1 Distributions of different number

Let's now consider some distributions regarding data collected during Vinitaly's 35th edition, held in Verona in 2001. At this event, a test was performed on consumers, called *Grappa & C. Tasting*. In general, each product had a different tasting frequency, because the number of people tasting one or other product varied.
We selected five *grappe* which showed differences for the "taste-olfactory" (*to*) descriptor on a scale ranking from 1 (*very bad*) to 6 (*very good*). The distributions, listed according to the increasing value of $\bar{s}$ (μ), are shown in Table 4. Since frequencies have a different total *n*, we also provide the *BM* in percentage form to make comparative analysis more effective.
G_1(*to*) appears to have the "worst" distribution since it shows the smallest μ (together with the corresponding *BM*) and the highest variability index (60% of the maximum). Instead, G_5(*to*) has the "best" distribution, showing the highest mean and the smallest variability index (22,22% of

the maximum). Similar arguments can also be used for other distributions. The main conclusion is that the order established by the mean is not altered by the variability index.

Table 4. Frequency distributions and *BM* of the scores given by *n* different tasters for five *grappe* according to the taste-olfactory descriptor and correspondent indexes

Grappe	$G_1(to)$			$G_2(to)$			$G_3(to)$			$G_4(to)$			$G_5(to)$		
Scores	n_s	η	η%	n_s	η	η%	n_s	η	η%	n_s	η	η%	n_s	η	η%
1	2	0	0	2	0	0	0	0	0	0	0	0	0	0	0
2	2	0	0	2	0	0	1	0	0	1	0	0	0	0	0
3	1	10	100	6	11	73	7	5	31	1	0	0	1	0	0
4	4	0	0	1	4	27	5	11	69	7	4	19	2	0	0
5	1	0	0	3	0	0	2	0	0	4	17	81	10	17	100
6	0	0	0	1	0	0	1	0	0	8	0	0	4	0	0
n	**10**			**15**			**16**			**21**			**17**		
$\bar{s}\ (\mu)$	**3**			**3,2667**			**3,6875**			**4,8095**			**5**		
S_η	**1,2**			**0,8**			**0,5**			**0,7619**			**0,4706**		
S_η^*	$\frac{6-2}{2}=\mathbf{2}$			$\frac{6-2}{2}\times\frac{16}{15}=\mathbf{2{,}133}$			$\frac{6-2}{2}=\mathbf{2}$			$\frac{6-2}{2}\times\frac{22}{21}=\mathbf{2{,}095}$			$\frac{6-2}{2}\times\frac{18}{17}=\mathbf{2{,}118}$		
$S_\eta/S_\eta^*\%$	**60%**			**37,51%**			**25%**			**36,36%**			**22,22%**		

Finally, we investigate what happens when the distribution $S_1(to)$ is observed instead of the score distribution $G_1(to)$ (see Table 5 with $G_2(to)$).

Table 5. Comparison of score distributions $S_1(to)$ and $G_2(to)$

Scores	$S_1(to)$		$G_2(to)$	
	n_s	η	n_s	η
1	0	0	2	0
2	1	0	2	0
3	6	8	6	11
4	3	2	1	4
5	0	0	3	0
6	0	0	1	0
N	**10**		**15**	
$\bar{s}\ (\mu)$	**3,2**		**3,2667**	
S_η	**0,2**		**0,8**	
S_η^*	**2**		**2,133**	
$S_\eta/S_\eta^*\%$	**10%**		**37,51%**	

In this situation, the mean of distribution $S_1(to)$ is smaller than the mean of $G_2(to)$ but the difference is minimal (3,2 *vs* 3,2667). On the contrary, the variability index of $S_1(to)$ is appreciably smaller than $G_2(to)$ (37,51% of the theoretical maximum for the first *vs* 10% of the second). In this case, the order established by μ could be modified, judging the first distribution as "the best one"; where in fact a smaller variability in the judgements is recognised, despite its slightly smaller mean.

4. Conclusions

Recently, Walter Maffenini and Michele Zenga [5] introduced a new synthesis for ordinal variables: the Bipolar Mean. This is a distribution that concentrates the total size *n* on one of the *k* categories of the variable or, at most, on two consecutive categories and it is consistent with the principle of statistical dominance. Later, Maffenini and Mariangela Zenga [6] extended the bipolar mean to discrete variables while defining a variability measure: the Mean Deviation about the Bipolar Mean. This measure can also be computed for ordinal qualitative variables when their categories are expressed as scores on a rating scale.

The mean deviation about the bipolar mean is an absolute index of variability and, with the aim of comparison, it is interesting to set its maximum value. Brentari *et al*. [3] e [4] defined the maximum variability distribution and its bipolar mean. These measures are useful in the framework of sensory analysis where score distributions that synthesize the evaluations of *n* subjects relating to certain characteristics of different products are compared.

In this study, we applied the bipolar mean and its dispersion measure for evaluating certain types of food. In doing so, we highlight the summarizing feature of this index. The distributions used in the empirical analysis belong to two groups: the first contains the same number of cases while the second contains a different number of cases. We found that:

- two (or more) distributions with the same arithmetic and bipolar mean are equivalent (according to the order established by these factors). If the correspondent variability measures have different values, we recognize the superiority of the less-variable distribution, since it shows less “dispersion” among the judgments made;
- two (or more) distributions with roughly the same mean values can be ranked in a different way with respect to the bipolar mean when their variability measures are significantly different.

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