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APPROXIMATE EVALUATION OF CUMULATIVE DISTRIBUTION FUNCTION OF CENTRAL SAMPLING DISTRIBUTIONS: A REVIEW

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Abstract: *Cumulative distribution functions of some sampling distributions do not have closed form representation. Hence an approximate formula can be of immense use. In this paper, we review the development of literature of approximations to cumulative distribution function of some popular central sampling distributions.*

Keywords: *Cumulative distribution function, approximation, t distribution, chi-square distribution, F distribution.*

1. Introduction

Statistical computations frequently involve the cumulative distribution function (cdf) or its complement, the survival function. It is well known that the cdf of many continuous distributions do not have closed form representations. Common examples are Student t distribution, chi-square, F and normal. Since cdfs do not have closed form representation, their inverses (percentage points or quantiles) also do not have closed form representation.

In view of this handicap, often one has to refer to standard tables for its evaluation. However such tables are cumbersome and do not always suffice. Many commonly used application software such as Ms-Excel as well as specific statistical software like SPSS, STATA provide numerical evaluation of cdfs. Such software has fixed subroutines dealing with specific techniques. However in many statistical computations, the cdfs are required in a manner that is beyond what is available in the application software. When such circumstances arise, academicians and practitioners often code programs in different programming languages (FORTRAN, C etc.) as per their requirement. Presently libraries of such programming languages

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do not offer any in built subroutine or function to compute cdf's. Hence, an approximate formula is used in coding. For most common distributions, many standard approximations are now in use. There is a vast amount of literature on the subject of approximating cdf's of some continuous distributions. In this short review, we shall deal with approximation to the cdf's of the popular sampling distributions viz. Student t, chi square and F distributions. The purpose of this review is to lay down the development of literature with an eye on how they evolved.

2. Approaches to approximating cdf's

While reviewing literature, we have come across three broad approaches for approximate evaluation of the cdf of some well known continuous univariate distributions. Broadly, these can be classified as follows:

a) Construction of approximate formula.
b) Use of approximations of other distributions to the distribution whose cdf needs to be computed.
c) Construction of bounds.

Our object here is not to discuss which of these approaches work best. We assume that all the approaches work and all lead to results which can reasonably be considered as approximations to cdf. The review shall touch upon all the three approaches.

3. Notations

By cdf of t, chi-square and F distribution we shall mean the following:

$$G(t/\nu)=\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\,\Gamma(\nu/2)}\int_{-\infty}^{t}\left(1+\frac{t^2}{\nu}\right)^{-\left(\frac{\nu+1}{2}\right)}dt.$$

$$G(x/n)=\frac{1}{2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)}\int_{0}^{x}e^{-\frac{x}{2}}x^{\frac{n}{2}-1}dx.$$

$$G(f/m,n)=\frac{1}{\beta\left(\frac{m}{2},\frac{n}{2}\right)}\int_{0}^{f}\frac{(m/n)^{\frac{m}{2}}f^{\frac{m}{2}-1}}{\left(1+\frac{m}{n}f\right)^{\frac{m+n}{2}}}.$$

where $G(t/\nu)$, $G(x/n)$ and $G(f/m,n)$ are the cumulative distribution functions of Student t, chi square and F distributions respectively. Here we have considered t to be a t variate with ν

degrees of freedom, x to be a chi square variate with n degrees of freedom and f to be a F variate with (m, n) degrees of freedom.

4. Review of approximations to cdf of t distribution

4.1 Construction of approximate formula

A vast amount of literature exists for this category of approximation. Fisher [13] gave a direct expansion of the probability density, and hence of cdf, as a series in inverse powers of degrees of freedom. Much later, Ling [31] performed calculation on tail probability approximation given by Fisher [13]. Elfving [9] approximation, relying partly on standard normal cdf have errors less than $0.5\nu^{-2}$ times the exact cdf of t distribution, for all values of t. Kramer [27] obtained approximations for area lying between 0 to t and found that his approximations differs from the tabulated probabilities by no more than one unit in the third decimal place for $3 \le \nu \le 120$. Even for 1 degrees of freedom and for different ranges of t, his approximations are accurate to three decimal places and for 2 degrees of freedom, he found an exact expression. Pinkham and Wilk [44] suggested an expansion for calculating extreme tail probabilities of the t distribution and showed that good approximations can be obtained using only three terms in the expansion. In Zelen and Severo [58], an approximation with ten numerical constants is stated for $\nu \le 5$ but t large. Another approximation useful for large degrees of freedom can also be found in Zelen and Severo [58], but the approximation is in terms of normal cdf and hence comes under our second category of approximation to cdf of t distribution. Gentleman and Jenkins [16] found the most satisfactory approximation to the tail probability for more than five degrees of freedom to be a fifth degree polynomial raised to the minus eighth power. This approximation gives five decimal place accuracy for $\nu > 5$. Andrews [2] obtained a general method for approximation of tail areas of some common distributions such as standard normal, t, chi-square and F distributions and tabulated the relative error of tail probability approximation of t distribution in Table 2A of his paper. Sinclair [50] provided tail probability approximation for large values of t. The difference between the true value of tail probability and Sinclair's approximation is of order $t^{-(\nu+4)}$. Comparisons between Pinkham and Wilk [44], Mickey [34] and Sinclair [50] are provided in Sinclair [50].

4.2 Use of normal distribution for approximating cdf of t distribution

Numerous normal approximations to t distribution exist in literature. Two simple normal approximations are given in Abramowitz and Stegun [1], but unless degrees of freedom are fairly large, neither approximation is recommended. Hotelling and Frankel [23] sought to find a function of t with a distribution that is well approximated by the unit normal distribution. The leading terms of their series are a Cornish-Fisher form of expansion, while the successive terms rapidly becomes more complicated. Quenouille [46] suggested a normalizing transformation using an inverse sinh transformation, which is fairly good provided degrees of freedom is not too small. Anscombe [3] suggested a similar normalizing transformation using an inverse sinh transformation which is suitable for smaller values of degrees of freedom. Cornish [7] reported Hill's [19] normal approximation of Cornish-Fisher type of expansion. Some other normal approximations of Cornish – Fisher type of expansions are given by Hill [20, 21, 22]. Among

Hill's approximations, modified third term approximation given by Hill [20] is the best in terms of maximum absolute error. Wallace [54] obtained two good approximations for normal deviate, where one of the approximations is simpler than the other. The simpler approximation seems to be within 0.02 of normal deviate when $\left(t^2/v\right) < 5$ and the other, slightly complicated one seems to be within 0.02 of normal deviate for a wide range of values of t. Peizer and Pratt [42] proposed two normal approximations where one is slightly simpler than the other. Ling [31] performed calculation of tail probability approximations given by Peizer and Pratt [42] and tabulated the maximum absolute errors in Table 2 of his paper for selected degrees of freedom, over the range of the random variable for which the tail probability is between 0.0001 and 0.4999. Mickey [34] normalizing transformation is a modified Chu [6] transformation. Bailey [4] proposed two normalizing transformation of a Student's t variate. One of the transformations is accurate to $0\left(v^{-3}\right)$ locally at the standard normal variate $z = 1.9469$, while the other is very accurate locally at a prescribed deviate of the standard normal distribution. For the prescribed deviate $z_c = 1.9600$, 2.5758 and 3.2905, very high accuracy upto $0\left(v^{-3}\right)$ is achieved and for $z_c = 2.3276$, the accuracy improves to $0\left(v^{-4}\right)$. Li and De Moor [28] proposed a corrected normal approximation for Student's t cdf for $v > 3$ .This approximation involve shrinking factor and has a theoretical error $0\left(v^{-2}\right)$ uniformly in t. Even for very small degrees of freedom, it can give satisfactory accuracy.

4.3 Construction of bounds

Chu [6] obtained upper and lower bounds for cdf of t distribution in terms of normal cdf where the proportional error in using the normal cdf as an approximation to t cdf is less than v^{-1} for all t and for all $v \geq 8$. Unlike Chu [6] where bounds are obtained for cdf of t distribution, Wallace [54] obtained bounds on normal deviate. Wallace bounds show that $\left[v \log\left\{1 + \left(t^2/v\right)\right\}\right]^{\frac{1}{2}}$ as an approximation to normal deviate has an absolute error not exceeding $0.368v^{-\frac{1}{2}}$ and a relative error not exceeding $0.254/v$. Soms [52] obtained bounds for the ratio of the upper tail area of the normal distribution to the upper tail area of the t distribution. He showed that the lower bound is valid for all $v \geq 1$ and the upper bound for $v \geq 2$. Similar bounds were obtained by Soms [51], though this earlier result of Soms is good in terms of simplicity.

4.4 Comparison of approximations

Ling [31] examined the numerical accuracy of some computational formula for the tail areas of t distribution in terms of maximum absolute error. He compared tail probability approximations given by Cornish-Fisher (Johnson and Kotz [24], p. 102, eq. 10), Gentleman and Jenkins [16], Peizer and Pratt I [42] and Peizer and Pratt II [42], Wallace [54] etc. and tabulated the maximum absolute error in Table 2 of his paper for selected values of degrees of freedom $v, 5 \leq v \leq 120$, over the range of the random variable for which the tail probability is between 0.0001 and 0.4999. According to Ling, Cornish-Fisher approximation, Gentleman and Jenkins approximation , Peizer and Pratt II approximation and Wallace approximation are the most accurate approximations, but one prefer to use the latter two (Peizer and Pratt II and Wallace, respectively) simply because they require much shorter mathematical expressions. Lozy [32] also compared numerous t approximations such as Gentleman and Jenkins [16], Peizer and Pratt II [42], Cornish-Fisher (Johnson and Kotz [24], p. 102, eq. 10), and various approximations of Hill (2-term, 3 term and 3 term modified). It is clear from Figure 1 of Lozy [32] paper that the two and three term Hill approximations are considerably better than the other approximations. The Hill approximation is more accurate and much faster than the Cornish-Fisher approximation. The Gentleman-Jenkins and Hill approximations are the only ones to give five correct decimal places for a small number of degrees of freedom, but Lozy [32] advocated the use of Hill two term approximation over the complicated Gentleman and Jenkins one. He also pointed out two term Hill approximation gives an accuracy of five decimals for eight degrees of freedom, as opposed to about 45 for the Peizer-Pratt and Wallace approximations. In fact the former approximation is no more complicated than any of the latter approximations. Hence two term Hill approximation is preferable to Peizer-Pratt and Wallace approximations.

5. Review of approximations to cdf of chi-square distribution

5.1 Construction of approximate formula

Gray, Thompson and McWilliams [18] developed a simple approximation for $Q(x/n)$ where $Q(x/n)$ is the area under the right tail of the chi square distribution. The accuracy of approximation has been computed for $n = 1$, 2, 5, 10, 20, 30, 100, 250, 500, using values of x which yield $Q(x/n)$=0.1, 0.05 and 0.01 in Table 1 of Gray, Thompson and McWilliams [18] paper. This approximation gives an accuracy of three decimal places when $Q(x/n)$ is of order 0.1, even for degrees of freedom n as small as 2. Andrews [2] obtained a general method for approximation of tail areas of some common distributions such as standard normal, t, chi-square and F distributions and tabulated the relative error of tail probability approximation of chi-square distribution in Table 2B of his paper. Lin [30] proposed an approximation to the cumulative chi-square distribution and its inverse. He developed a direct approximation to the cumulative chi-square distribution through a square root transformation and tabulated the absolute relative error of this approximation in Table 1 of his paper.

5.2 Use of normal approximations

To develop approximate formula of cdf of chi-square distribution, another approach is to normalize the chi-square variate x to a standard variate z such that $\Phi(z) \approx G(x/n)$ where $\Phi(z)$ is the cdf of standard normal distribution and $G(x/n)$ is the cdf of chi-square distribution. Thus, the chi-square approximation consists in determining z corresponding to x by an approximate transformation and then evaluating the cdf $G(x/n)$. Numerous approximations for transforming a chi-square variate into a standard normal variate have been proposed in literature. Standardized chi-square distribution with n d.f. tends to the unit normal distribution as $n \to \infty$.Thus the simple normal approximation follows from the Central Limit Theorem and is given in Rao ([47], pp 222). This approximation is not very accurate unless degrees of freedom n are very large. Fisher (Rao[47], pp 222) took $\sqrt{2x}$ to be normally distributed about $\sqrt{2n-1}$ with unit standard deviation and suggested a normal approximation for cdf of chi-square distribution. This approximation is an improvement over that of simple normal approximation and is satisfactory if

$n > 100$. Freeman-Turkey ([14], [15]) presented another normal approximation, very similar to that of Fisher's approximation. Wilson Hilferty [55] approximation is better than Fisher (Rao [47], pp 222) approximation. Cornish-Fisher [8] presented a normal approximation to cdf of chi-square distribution. The simplest approximation – the log transformation approximation given by Wishart [56] is a normal approximation. Some other approximations are also given by Wishart [56]. Wallace [54] presented a normal approximation to the cdf of chi-square distribution, but this approximation is good for the right tail $(x > n)$ only. Severo and Zelen [49] presented normal approximation to the chi-square and non central F probability functions. The chi square approximation proposed by Severo and Zelen [49] is an improved Wilson Hilferty [55] approximation. Peizer and Pratt [42] obtained normal approximations to the cdf of chi-square distribution. Numerical results were obtained for this approximation by Narula and Li [36] and Ling [31] in terms of percentage error and maximum absolute error respectively. Narula and Li [36] and Ling [31] obtained results for numerous normal approximations and is discussed in detail in Section 5.4. Hill [19] presented a modified Cornish-Fisher type of expansion of chi-squared, using a function of chi-squared and the degrees of freedom first used by Wallace [54]. This normal approximation due to Hill [19] gives accuracy up to five decimal places for $n > 20$ (in fact, to eight decimal places if terms up to order n^{-4} are retained). Canal [5] proposed an accurate normal approximation for the cdf of chi-square distribution and performed numerical assessment of this approximation in terms of maximum absolute error for $1 \le n \le 1000$.

5.3 Construction of bounds

Wallace [54] obtained bounds on normal approximations to chi-square distribution. He considered the formula $w(\chi^2) = \left[\chi^2 - n - n\log\left(\chi^2/n\right)\right]^{\frac{1}{2}}$ for converting upper tail values of chi-square variate with n degrees of freedom to normal variate. He obtained bounds for tail probability and on normal deviate. Bounds on normal deviate, exact normal deviate and the Wilson Hilferty approximate deviate are illustrated in Table 2 of Wallace [54] paper for $n = 8$ and selected values of x. From the comparison it has been found that the Wilson Hilferty approximation is much superior to the bounds as approximations except in the extreme tail area.

5.4 Comparison of chi-square approximations

Narula and Li [36] compared the computational effort and accuracy of various chi-squared approximations. The accuracy of the approximations was compared in terms of percentage error. The number of arithmetic and special function operations required was used as a criterion for judging the degree of computational effort required. Narula and Li compared some approximations such as Peizer-Pratt, Severo-Zelen, Wilson-Hilferty, Cornish-Fisher, Freeman-Tukey, log transformation etc. and tabulated the computational effort of these approximations in Table I of their paper. They found errors to be less than 3 percent for $n \ge 5$ and less than 0.3 percent for $n \ge 10$ for the Severo-Zelen and the Peizer- Pratt approximation. The Cornish-Fisher approximation has errors less than 1 percent for $n \ge 10$. The Wilson Hilferty approximation has errors less than 1 percent, the Fisher approximation has errors less than 3 percent and Freeman-Tukey approximation has errors more than 10 percent for $n \ge 10$. Ling [31] examined the numerical accuracy of some computational formula for the tail areas of chi-square distribution in terms of maximum absolute error. He compared four normal approximations such as asymptotic approximation, Fisher's approximation, Peizer-Pratt approximation and Wilson-Hilferty approximation. Table 3 of Ling [31] show maximum absolute error of the tail areas of these four normal approximations for selected values of degrees of freedom n, $5 \le n \le 240$. The Asymptotic approximation is the least accurate. Though Fisher's approximation and Wilson-Hilferty approximation are the best known of the four formulas but they are less accurate than that of Peizer-Pratt approximation. Lozy [32] compared Hill [19] approximation to that of Peizer and Pratt [42] approximation and found that Hill's approximation performs better than Peizer and Pratt approximation, but this greater accuracy is achieved at the cost of involving considerably more mathematical operations.

6. Review of approximations of cdf of F distribution

6.1 Construction of approximate formulas

Numerous F approximations exist in literature. But most of the approximations developed are either normal approximations to F distribution or chi-square approximations to F distribution. Hence under our category of approximation "Construction of approximate formula" not much amount of literature is available. Andrews [2] approximation comes under this category of approximation. Andrews [2] obtained a general method for approximation of tail areas of some common distributions such as standard normal, t, chi-square and F distributions and tabulated the relative error of tail probability approximation of F distribution in Table 2C of his paper.

6.2 Use of approximations of other distributions to approximate cdf of F distribution

It is well that the distribution of $u = \frac{1}{2}\log F$ is more nearly normal than that of F. For large values of m and n, the distribution of u may be approximated by a normal distribution with expected value $\frac{1}{2}\left(n^{-1} - m^{-1}\right)$ and variance $\frac{1}{2}\left(m^{-1} + n^{-1}\right)$. Fisher ([11], [12]) method reported in Johnson et al. [25] yields a normal approximation to the F distribution. This approximation is good when m and n are both large. Viveros [53] presented numerical result for Fisher approximation in Table III of his paper considering F to follow $F(2m, 2n)$ distribution. Paulson [40] obtained an approximate normalization of the F distribution. He selected a modified statistic U, a function of F so that U tends to have a nearly normal distribution with zero mean and unit variance. For small values of $n \le 3$, Kelly [26] recommended replacing U of Paulson's approximation by $U' = U\left(1 + 0.08U^4 n^{-3}\right)$. Pinkham [43] obtained an approximate normalization of the F distribution. Pinkham approximation is also reported in Abramowitz and Stegun ([1], eq. 26.6.14). The asymptotic approximation based on the asymptotic normality of the F distribution is reported in Abramowitz and Stegun ([1], eq. 26.6.13). Viveros [53] performed numerical assessment for Asymptotic approximation in Table II of his paper, considering F to follow $F(2m, 2n)$ distribution. Peizer and Pratt [42] obtained normal approximations to the cdf of F distribution. Comparisons of some normal approximations to tail probability of F distribution in terms of maximum absolute errors are shown in Ling [31] paper. Mudholkar and Chaubey [35] and Ojo ([38], [39]) also presented approximations to cdf of F distribution. [1]The approximations of Mudholkar and Chaubey [35] only on exceptional cases improve over Paulson approximation. The approximation of Ojo ([38], [39]) is a second order improvement over Fisher's approximation. Viveros [53] proposed power transformation of the F distribution, yielding simple normal approximation for both probabilities and quantiles of the distribution. Viveros employed the method of Ling [31] to assess the numerical performance of his approximation and tabulated the maximum absolute error for a selection of values of $2m$ and $2n$ in the range 10 to 240 over the range of the random variable for which the tail probability is between 0.0001 and 0.9999 in Table 1 of his paper. He compared his approximation with asymptotic approximation, Fisher approximation, Pinkham approximation, Paulson approximation etc. considering $F(2m, 2n)$ distribution and concluded that for the most part Paulson approximation performs better than the proposed approximation given by him. Ferreira [10] proposed normal shrinking factor approximation (SFA) for the cdf of F distribution. Ferreira compared his approximation with some other approximations such as ordinary normal (or asymptotic approximation), Paulson's approximation, ordinary chi-square approximation, Schefee and Tukey, Li and Martin approximation etc. In comparison with the normality based approximations, the accuracy of Ferreira approximation is greater than that ordinary normal and is practically the same as that of Paulson's approximation. When the ratio (n/m) is small for large m, the accuracy of normal SFA is greater. The normal SFA presents greater accuracy than the ordinary chi-square except for small m and large (n/m). The normal SFA exhibits larger accuracy than Scheffe-Tukey approximation when the ratio (n/m) is small, even for small value of m. The advantage of normal SFA given by Ferreira [10] over Li and Martin SFA [29] in the sense that normal cdf can easily be obtained than that of chi-square. Wong [57] obtained a simple normal approximation for the cdf of the F distribution via a general version of the modified signed log-likelihood ratio statistic. Even when the degrees of freedom are small, Wong's approximation exhibits remarkable accuracy.

The chi-square is often used as an asymptotic approximation to the F distribution. (Johnson et al. [25]). The ordinary chi square approximation is good for large n and fixed m. Scheffe and Tukey [48] proposed an approximation to improve the ordinary chi-square approximation, which is good for large n and fixed m. Li and Martin [29] proposed a Shrinking Factor Approximation (SFA) to F distribution to improve the Scheffe-Tukey approximation. For $(n/m) \geq 3$, accuracy of Li and Martin's SFA is to the fourth decimal place for most values of m. The theoretical approximate error of the SFA is $O(1/n^2)$ uniformly over f.

The cdf of F distribution can be expressed as an incomplete beta function. When both degrees of freedom are even, the F distribution can be expressed as a sum of binomial probabilities. For atleast one odd degrees of freedom, George and Singh [17] obtained approximations to cdf of F distribution in terms of binomial probabilities. Similar approximation of this type was earlier given by Mantel [33]. George and Singh [17] found that when both m and n are odd, their approximation gives superior results, especially in the tails of the distribution, but when one of m and n is odd, Mantel's approximation is slightly better.

6.3 Comparison of F approximations

Ling [31] examined the numerical accuracy of some computational formula for the tail areas of F distribution in terms of maximum absolute error. He compared tail probability approximation given by three normal approximations such as Asymptotic Approximation (Abramowitz and

Stegun [1], eq. 26.6.14), Paulson [40] approximation, Peizer-Pratt ([42], p.1420) approximation and tabulated the maximum absolute errors of the tail areas of these three normal approximations in Tables 4, 5 and 6 respectively for selected values of degrees of freedom m and n over the range of variable for which the tail probability is between 0.0001 and 0.9999. He concluded that the Peizer Pratt approximation is better than the Paulson approximation (generally by one to two orders of magnitude), which in turn is better than the asymptotic approximation by about one order of magnitude. Lozy [32] compared the accuracy of single precision versions of the Peizer-Pratt approximation to F using the "direct" and "best" codings, together with results from the double precision computations of Ling. He found that with careful coding four decimal places can be obtained for F probabilities with reasonable computational effort.

7. Conclusion

We have reviewed a number of approximate formulas and bounds. As mentioned earlier, their accuracy and ease of use varies. How does one make a choice in such circumstances? We place below some of the essential characteristics:

1. It must be easily programmable.
2. It must not be too cumbersome computationally.
3. It must not be grossly inaccurate over some ranges of values of the random variables.
4. The error in approximation is small.
5. Overestimates are preferable to underestimates.
6. Simplicity.
7. The expression to be evaluated should be easy to remember and very simple with few key strokes.
8. The approximation may be easily inverted, i.e., it is easy to find the cut point if the area is known.

Many of these have been advocated by Ling [31] and Norton [37]. At the present point of time, because of availability of computers at a reasonably cheap price, properties 2, 6 and 7 may not be so relevant though they remain desirable. It is pertinent to note here that we could not locate a single formula, which satisfies all the above desirable characteristics. The choice of an approximate formula or bound therefore has to be made on the basis of one's individual choice and perception of importance of the desirable characteristics.

The domain of this review is central sampling distribution. We are aware that considerable literature is available on approximate evaluation of cdf of non central sampling distribution. This is relevant in view of the fact that central sampling distributions can be obtained as particular case of non central sampling distributions. We have however omitted the non central sampling distributions so as to improve the focus of this paper. Perhaps this is a limitation of this work.

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