A nonlinear generalization of Arbitrage Pricing Theory

Gerarda Tessitore
Second University of Naples
gerarda.tessitore@unina2.it

Rosaria Lombardo
Second University of Naples
rosaria.lombardo@unina2.it

Francesco Gangi
Second University of Naples
francesco.gangi@unina2.it

Abstract: *Arbitrage Pricing Theory* (*APT*) *leads good estimates of expected utility stock returns by means of k factors. Notwithstanding initial skepticism the idea of using multiple risk factors to explain the relationship between expected return and asset risk has been winning. In literature the APT has been seen as a generalization of single risk factor approach of Capital Asset Pricing Model (CAPM). The APT provides a better indication of asset risk and a better estimate of expected return than CAPM does. In this paper we propose a generalization of APT to non-linear case. In order to study the relationships which occur between return and multiple risk factors, we propose non-linear principal components. To find justifications for embracing a more complicated model than traditional APT we evaluate the consistency of results by known real data.*

Keywords: Arbitrage Price Theory, Risk factors, Non-linear Principal Component Analysis

1. Introduction

Empirical evidence indicates that the simple one period capital asset pricing model (CAPM; Sharpe, 1964; Lintner, 1965; Black, 1972) is inconsistent and alternative models of capital market equilibrium should be investigated. The arbitrage pricing theory (APT, Ross 1976) is built on the assumption of no arbitrage opportunities in the capital market and a linear relationship between actual returns on K common factors; the expected returns should be linearly related to the weights of the common factors in the assumed linear process. APT shows to be a plausible alternative to the simple one-factor CAPM as it explains empirical anomalies; in this respect Reinganum, 1981, shows that when portfolios are formed on the base of firm size, small firms systematically experience average rates of return greater than those of large firms. Roll and Ross (1980) claim to find empirically at least three or four factors, even if they do not offer an economic interpretation of those factors. Bower D. Bower R. & Louge D. (1984) show how APT may lead to different and better estimates of expected return and provide better indication of asset risk than CAPM, particularly in the case of utility stock returns, while offering a systematic link between expected return and the return generating process. CAPM is weak as it is based on Efficient Market Hypothesis, which means: transparency; no transaction costs; no significant restriction to investment (such as capital rationing); investors rational behavior and expectations. These assumptions imply that the market is not affected by imperfections, such as the absence of fiscal arbitrage opportunity, that is the most important. These assumptions applied to the real market discount a paradox: the market efficiency hypothesis works properly only if a sufficient number of investors don't recognize this efficiency!
On the other side, APT construction assumes that the price of a security is linearly related to more than one variable. These variables can be divided in two groups: macro-economics and firm-

specific variables; moreover the second group of variables can be diversified by efficient portfolios. However the APT main problem is that the theory in itself doesn't provide indications on the variables or explanations why these factor are relevant; simply it assumes and proves a relationship between share returns and these variables. So, in literature various versions of APT variables specification have been proposed. Van Horne considers the dividend, the capital dimension expressed by the market capitalization, the industry sector, the leverage and the P/E ratio. Moreover Fama and French (Fama E., Frenck K., 1993) use three main variables: the market return minus the free risk rate (*market variable*); the return on small stocks minus the return on large stocks (*size variable*); the return on high book-to-market stocks minus the return on low book-to market stocks (*book-to-market stocks*). Instead, Elton, Gruber and Mei (1994) enlarge the model to five variables: the long term free risk return minus the 30 days free risk return; the free risk return variation; the exchange rate of dollar respect to other currencies; the gross domestic product variation; the rate inflation variation. As shown, based on economic intuition, researchers tend to add new variables and test new statistical applications (Huberma G.,Wang Z., 2005). However, the essential consideration is that the different versions of the APT model have common denominators represented by a fundamental empirical approach and a more complex collected data to calculate the sensitivity of the price of the security to more than one variable. Most theoretical research has been based on perfect efficient and frictionless markets, but the pricing predictions, thus derived, sometimes stray from their fundamental values (or from the values of the replicating portfolio). Recent literature on behavioral finance provides some answers to pricing patterns incorporating frictions, such as transactions costs, short sale constraints, and tax considerations, into rational general equilibrium models.

Risk variables are often nonlinearly each-other related; that's why we propose to use nonlinear principal components (Gifi, 1990) instead of linear factors. The empirical risk variables are transformed by B-spline functions in order to catch the nonlinear risk effects, and subsequently the principal component analysis is performed on the nonlinearly transformed variables. The out-coming component scores and loadings are used in the study of the dependence between capital return and risk factors. In nonlinear APT risk factors are given by nonlinear principal scores and factor loading by nonlinear principal loadings. This research investigates empirically whether nonlinear APT conveys more information than APT does, while exploring more deeply the relationship among variables, in this case not forced to be liner. After a brief introduction to the arbitrage pricing theory, the nonlinear APT and its method of estimation are discussed.

2. From CAPM to APT and method of Estimation

CAPM decomposes a portfolio's risk into systematic and specific risk; the expected return of an asset (or derivative) equals the risk-less return plus a measure of the assets non-diversifiable risk ("beta") times the market-wide risk premium (excess expected return of the market portfolio over the risk-less return). The logic measurement of Capital Asset Pricing Model (CAPM) is simple and attractive and the systematic risk can be estimated by applying the market model: $r_t = \alpha_0 + \beta_i r_M + \hat{\varepsilon}_i$, where r_i is the estimated return on asset *i*, r_M is the return on the market portfolio in the same period, β_i is the estimate of the systematic risk and measures the sensitivity of r_i to the market index variation, α_0 is the risk free rate of return, ε_i is an error term with zero mean representing the residual risk.

An important and more useful version of CAPM is represented by the "*correct beta model*", in which the single non-company factor is corrected by two specific company factors: *leverage* and *tax rate*. In this case the formula is more representative of the effective relationship between risk and expected return of a specific security; and the beta is represented as follows: $\beta\left[\left(1+\frac{D}{E}\right)(1-T_c)\right]$, where *D*/*E* is the leverage corrected by the tax-shield $(1-T_c)$ and *Tc* is the tax rate (Ross S.A.,

Westerfield R.W., Jaffe J.F., 1997). It is evident that the model gives a measure of the fundamental relationship risk/return, like modified by company specific variables, while remaining connected to only one macro-variable; very near to the APT construction.
Furthermore, as well as CAPM, APT describes expected return as linear function of systematic risk but it reflects the possibility that there may be more than one systematic risk factor, macro and company specific factors. The arbitrage pricing theory is essentially based on the following three assumptions: a) capital markets are perfectly competitive; b) investors always prefer more wealth than less wealth; c) the stochastic process generating asset returns can be represented as K-factor model of the form:

$$r_i = r_F + \sum_{j=1}^{K} b_{ij} f_j + \hat{\varepsilon}_i \qquad \text{with } i=1,\ldots,n \qquad (1)$$

where n is the number of assets; r_i is the return on asset i; r_F is the risk free asset return; f_j is a common factor, with zero mean, that influences returns on all assets; b_{ij} is the reaction of capital asset return i for a unit change in component factor f_j; finally ε_i is by assumption an idiosyncratic effect on the i-th asset return, which has zero mean and is completely diversifiable in large portfolios. The economic argument of APT is that in equilibrium the return on a zero-investment, zero-systematic risk portfolio is zero, as long as the idiosyncratic effects vanish in a large portfolio. On this basis expected return on any asset i come to be expressed as: $E(r_i) = \lambda_0 + \lambda_1 b_1 + ... + \lambda_k b_{iK}$, where the term λ_0 represents the expected return on an asset with zero systematic risk (with $b_{01} = b_{02} = ... = b_{0K}$); the weights $\lambda_1, \lambda_2, ..., \lambda_k$ can be interpreted as factor risk premia, and the b_i's reflect the price relationship between the risk premia and asset i.
The stochastic process in (1) permits to estimate the b_{ij} coefficients by the use of factor analysis (see Harman, 1976), and the b_{ij} coefficients are referred to as factor loadings; the K vectors $\mathbf{b}_1, \ldots, \mathbf{b}_k$ are the factor loadings vectors, each of dimension Nx1. In practice the reaction coefficient that characterize an asset are estimated from a market model, that is from a linear factor model.

3. The Nonlinear APT and Method of Estimation

On the same three assumptions of arbitrage pricing theory, we propose the non linear extension of the above model essentially based on the linear extension of a factor model, i.e. Non-linear Principal Component Analysis (NL-PCA, Gifi, 1990).

3.1 General Factor Analysis and NL-PCA

To understand the strict relation between linear factor Analysis and NL-PCA we describe at first a linear factor model. By the least squares properties of Singular Value Decomposition (Eckart & Young, 1936) classical PCA seeks the following minimum: $\sum_{j=1}^{K} (\mathbf{x}_{\mathbf{P}j} - \mathbf{F}\mathbf{b}_j)^2$, where $\mathbf{X}_\mathbf{P}$ is the original data matrix (of order N,P, with P the number of variables, N the number of individuals) and $\mathbf{x}_{\mathbf{P}j}$ the generic column; $\mathbf{F}$ the component scores (object or factor scores) matrix of dimension N,P (with $P \geq K$); $\mathbf{b}_j$ the vector of weighting coefficients, called component or factor loadings, found by least squares regression. The P original variables represent underlying economic forces that primary influence the stock market, for example unanticipated inflation, changes in the expected level of industrial production, unanticipated movements in the interest rate term structure, and, of course, the return on the market portfolio, like CAPM model does.
The principal components can be seen as the least squares estimates in the linear factor model

$$\mathbf{x}_{\mathbf{P}j} = \mathbf{F}\mathbf{b}_j + \mathbf{e}_j \qquad \text{for } j=1,...,K \qquad (2)$$

where e_j are the uncorrelated errors with zero mean and constant variance. The re-formulation of (2) by Gifi (1990) leads to a generalization of metric PCA, also called Non-metric Component analysis or NL-PCA or Smooth PCA. It consists of a PCA on a priori non linearly transformed variables by the use of an ALS algorithm with normalization constraints. Assume that we start with variables in deviation from the mean and normalized to unit length, define the correlation matrix then it is easy to see that the latent root and the latent vector solutions of a classical PCA relate to the principal component **F** of the transformed data matrix. In real world, individuals face two layers of risk in economy, first about the trading environment (state of information) and then about the realization of endowments and asset payoffs (state of nature). At time zero they face uncertainty about the market participants endowments and preferences, which translates into uncertainty about prices. The reaction coefficients that characterize an asset are estimated from a non-linear market model. Combining both the assumption that in equilibrium the return on a zero-investment, zero-systematic risk portfolio is zero, as long as the idiosyncratic effects vanish in a large portfolio, using the regression model on non-linear principal components, as the component loadings are based on non-linear *PCA*. Finally we evaluate the consistency of results comparing linear and non-linear APT by empirical evidence studies.

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