MULTILEVEL LINEAR MODELS ANALYSIS USING GENERALIZED MAXIMUM ENTROPY

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Abstract
This paper introduces the general multilevel models and discusses the generalized maximum entropy (GME) estimation method (Golan et al 1996) that may be used to fit such models. The proposed procedure is applied to two-level data generated in a simulation study. The GME estimates are compared with Goldstein's generalized least squares estimates. The comparisons are made by two criteria, bias and efficiency. We find that the estimates of the fixed effects and variance components are substantially and significantly biased using Goldstein's generalized Least Squares approach. However, the GME estimates are unbiased and consistent; we conclude that the GME approach is a recommended procedure to fit multilevel models.

Keywords: **Multilevel Models, Generalized Maximum Entropy, Simulation, Goldstein's Generalized Least Squares**.

1. Introduction

Multilevel linear models or random coefficients models are a type of mixed model with hierarchical data in away that each group at the higher level (e.g., school level) is assumed to have different regression slopes as well as different intercepts for purposes of predicting an individual-level of the dependent variable. Random coefficients model is illustrated by Bryk et al (1992) and Goldstein (1987). The two levels model can be expressed as:

Level 1: $$y_{ij} = B_{0j} + B_{1j} * X_{ij} + r_{ij} \qquad \begin{matrix} i = 1,2,....,n_j \\ j = 1,2,....,J \end{matrix} \qquad (1)$$

where B_{0j} represents the intercept of group j, B_{1j} represents the slope of variable X_1 of group j, and r_{ij} represents the residual for individual i within group j. Also, J is the largest number of levels, and n_j is the jth level sample size.

Level 2: $$\begin{aligned} B_{0j} &= \gamma_{00} + \gamma_{01} * W_j + U_{0j} \\ B_{1j} &= \gamma_{10} + \gamma_{11} * W_j + U_{1j} \end{aligned} \qquad (2)$$

Where γ_{00} and γ_{10} are intercepts, γ_{01} and γ_{11} represent slopes predicting B_{0j} and B_{1j} respectively from variable W_j and U_0 *and* U_1 are random errors.

The traditional estimation method used to estimate the parameters of model given in (1) and (2) is the iterative generalized least squares method; which is a sequential refinement procedure based on ordinary least square (OLS) estimation. The method has been described in detail by Goldstein (1986). Briefly, equations such as (1 and 2) are expressed in the usual general linear model form $Y = \beta X + \varepsilon$, then the OLS estimation method is used. For the two-level model, we fit the first level by using OLS, which leads to the following estimates

$$\hat{\beta}_j = (\mathbf{x}'_j \mathbf{x}_j)^{-1} \mathbf{x}'_j \mathbf{Y}_j$$

and its dispersion matrix is given by

$$Var(\hat{\beta}_j) = \mathbf{V}_j = \sigma^2 (\mathbf{x}'_j \mathbf{x}_j)^{-1}$$

Then we use these estimates to fit the second level; so the parameters in the second levels can be estimated by

$$\hat{\gamma} = \left(\sum \mathbf{W}'_j \mathbf{W}_j\right)^{-1} \sum \mathbf{W}' \hat{\beta}_j$$

The recent of this paper is organized as follows: section.2 discusses the GME estimation method. Section.3 introduces a simple mote Carlo simulation study to compare the performance of GME and the OLS methods. The last section included some concluding remarks.

2. Maximum Entropy for Random-Coefficients Model

In order to estimate the two level random coefficient model by using GME method we rewrite equations (1) and (2) by one equation as

$$y_{ij} = \gamma_{00} + \gamma_{01} * W_j + U_{0j} + \left(\gamma_{10} + \gamma_{11} * W_j + U_{1j}\right) * X_{ij} + r_{ij} \qquad \begin{array}{l} i = 1,2,\ldots, n_j \\ j = 1,2,\ldots, J \end{array} \tag{3}$$

In the new general model (3), there are three unknown parameters and three error terms need to be reparametrized to fit the generalized maximum entropy principles. These reparameterized parameters are given as

1) $\gamma_{00} = \sum_{r=1}^{R} a_r p_r$ where $p_r \in (0,1)$ and $\sum_{r=1}^{R} p_r = 1$
2) $\gamma_{10} = \sum_{b=1}^{B} z_b q_b$ where $q_b \in (0,1)$ and $\sum_{b=1}^{B} q_b = 1$
3) $\gamma_{01} = \sum_{k=1}^{K} c_k N_k$ where $N_k \in (0,1)$ and $\sum_{k=1}^{K} N_k = 1$
4) $\gamma_{11} = \sum_{s=1}^{S} d_s G_s$ where $G_s \in (0,1)$ and $\sum_{s=1}^{S} G_s = 1$

and the error terms as

1) $u_{1j} = \sum_{d=1}^{D} y_{dj} f_{dj}$ where $f_{dj} \in (0,1)$ and $\sum_{d=1}^{D} f_{dj} = 1$

2) $u_{0j} = \sum_{i=1}^{E} v_{ij} T_{ij}$ where $T_{ij} \in (0,1)$ and $\sum_{i=1}^{E} T_{ij} = 1$

3) $r_{ij} = \sum_{l=1}^{M} v^{*}_{lij} O_{lij}$ where $O_{lij} \in (0,1)$ and $\sum_{l=1}^{M} O_{lij} = 1$

where $j = 1,2,......J.$ *and* $i = 1,2,.......n_j$

Using these reparameterization expressions, the model can be rewritten as

$$y_{ij} = \sum_{r=1}^{R} a_r p_r + \left(\sum_{b=1}^{B} z_b q_b\right) x_{ij} + \left(\sum_{k=1}^{K} c_k N_k\right) w_j + \left(\sum_{s=1}^{S} d_s G_s\right) w_j x_{ij} + \left(\sum_{d=1}^{D} y_{dj} f_{dj}\right) x_{ij} + \sum_{i=1}^{E} v_{ij} T_{ij} + \sum_{l=1}^{M} v^{*}_{lij} O_{lij}$$

Therefore, the GME model for the two levels model can be expressed by the following nonlinear programming system:

$$Maximize\ H(p,q,N,G,f,T,O) = -\sum p \ln(p) - \sum q \ln(q) - \sum N \ln(N) - \sum G \ln(G) - \sum fLn(f) - \sum T \ln(T) - \sum O \ln(O)$$

Subject to

(1) $$y_{ij} = \sum_{r=1}^{R} a_r p_r + \left(\sum_{b=1}^{B} z_b q_b\right) x_{ij} + \left(\sum_{k=1}^{K} c_k N_k\right) w_j + \left(\sum_{s=1}^{S} d_s G_s\right) w_j x_{ij} + \left(\sum_{d=1}^{D} y_{dj} f_{dj}\right) x_{ij} + \sum_{i=1}^{E} v_{ij} T_{ij} + \sum_{l=1}^{M} v^{*}_{lij} O_{lij}$$

(2) $$\sum_{r=1}^{R} p_r = 1,\ \sum_{b=1}^{B} q_b = 1,\ \sum_{k=1}^{K} N_k = 1,\ \sum_{s=1}^{S} G_s = 1,\ \sum_{d=1}^{D} f_{dj} = 1,\ \sum_{i=1}^{E} T_{ij} = 1, \sum_{l=1}^{M} O_{lij} = 1$$

3. Simulation Study

For the purposes of the simulation study we considered the following balanced random slope model

$$\begin{aligned} y_{ij} &= \beta_j x_{ij} + e_{ij} \qquad & i = 1,2,...,n \\ \beta_j &= \gamma_0 + \gamma_1 w_j + u_j \ , & j = 1,2,....,J \end{aligned}$$

Then the compound model:

$$y_{ij} = (\gamma_0 + \gamma_1 w_j + u_j) x_{ij} + e_{ij}$$

The simulation study performed under the following assumptions:

1) Generate 1000 random sample of size n = 10, 20, 30 and 50 and number of intercepts J = 2.
2) The error e ~ N(0, 1), X~Exp(1), and set $\beta_1 = \beta_2 = 1$
3) The error u ~ N(0, 1), W~U(0,1) and set $\gamma_0 = 1$, $\gamma_1 = 1.5$

4) For GME estimator, we initial three support values for parameter in the interval [-10, 0, 10] and three support values for the error term selected in the interval [- 3 S, 0, 3 S] where S is the standard deviation of the dependent variable y.
5) The simulation results for estimating fixed effect parameters , are given in Table.1

The simulated bias and the efficiency are computed based on the following formulas:

$$Bias = \frac{\sum_{i=1}^{1000}(\hat{\gamma}_i - \gamma)}{1000} \text{ and } Eff = \frac{MSE(OLS)}{MSE(GME)}.$$

	$\hat{\gamma}_0$			$\hat{\gamma}_1$		
Sample size	*Bias (OLS)*	*Bias (GME)*	*Eff*	*Bias (OLS)*	*Bias (GME)*	*Eff*
10	0.0287	0.0110	1.75	0.0162	0.0021	1.31
20	0.0281	0.0094	1.52	0.0123	0.0015	1.30
30	0.0270	0.0095	1.58	-0.0113	0.0015	1.19
50	0.0205	0.0082	1.62	0.0222	0.0010	1.24

Table. 1 Monet Carlo Comparisons between OLS and GME for the Random coefficient model

4. Concluding Remarks

The simulation results demonstrated that GME estimates are superior and often closer to the parameter than the OLS estimates. For all sample sizes used in the simulation study, there is an advantage for using the GME estimator. Consequently, the GME estimator can be recommended for estimating the two level random coefficients parameters.

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