Convergence of discontinuous games and essential Nash equilibria

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Abstract: Let Y be a topological space of non-cooperative games and let F be the map defined on Y such that F(y) is the set of all Nash equilibria of a game y. We are interested in finding conditions on the games which guarantee the upper semicontinuity of the map F. This property of F is a first requirement in order to study the existence of a dense subset Z of Y such that any game y belonging to Z has the following stability property: any Nash equilibria of the game y can be approached by Nash equilibria of a net of games converging to y.

Keywords: Discontinuous non-cooperative games, better-reply secure games, pseudocontinuous functions, essential Nash equilibria.

1. Introduction.
Let S₁,...,Sₙ be non-empty sets and let f₁,...,fₙ be real valued functions defined on the Cartesian product of the sets S₁,...,Sₙ. The list of data y=(S₁,...,Sₙ, f₁,...,fₙ) is an n-player non-cooperative game: for any i ∈ [1,...,n], Sᵢ is the set of strategies of the player i and fᵢ is the payoff function. If player 1 chooses the strategy x₁, player 2 chooses the strategy x₂ and so on, the corresponding outcomes of the game are: f₁(x₁,x₂,...,xₙ) for player 1, f₂(x₁,x₂,...,xₙ) for player 2,..., fₙ(x₁,x₂,...,xₙ) for player n. A list of strategies x=(x₁,x₂,...,xₙ) is said to be a profile of strategies, and a profile of strategies x* is said to be a Nash equilibrium (see Nash (1950)) if, for any player i, fᵢ(x*)≥fᵢ(xᵢ,x*-i) for all xᵢ belonging to Sᵢ, where (x₁,x*-i) means (x₁,x₂,...,xᵢ-1,xᵢ,xᵢ+1,...,xₙ). Now, let Y be a set of non-cooperative n-player games endowed with a topology. Following Yu (1999), given a game y, we say that a Nash equilibrium x of the game y is essential if for any neighbourhood O of x there exists a neighbourhood N of y such that any game y’ belonging to N has at least a Nash equilibrium x’ which belongs to O; moreover, we say that the game y is essential if any Nash equilibrium of y is essential. As one can see, the idea of essentiality concerns a notion of stability under perturbations on the data. Before to explain this, let us recall some definitions in the setting of set-valued analysis. Let F be a set-valued function (also said a map) defined on Y with values in X. We say that F is upper semicontinuous (see, for example, Aliprantis and Border (1999)) at y∈Y if for any open set O containing F(y) there exists a neighbourhood N of y such that F(y’)⊆O for any y’∈Y; we say that F is lower semicontinuous at y if for any x∈F(y) and any open neighbourhood O of x there exists a neighbourhood N of y such that F(y’)∩O is non-empty for any y’ belonging to N. Now, if F is the map defined on a space of games Y such that F(y) is the set of Nash equilibria of y, then the game y is essential if and only if the map F is lower semicontinuous. A crucial aid is the following theorem, due to Fort (1950):

Theorem 1. Let F be a set-valued function defined on Baire space Y with non-empty and compact values in a metric space X. If F is upper semicontinuous at any point of Y, then there exists a subset Z of Y which is dense and such that F is also lower semicontinuous at any point of Z.

In light of this theorem, if the topological space of games Y is a Baire space (for the definition of Baire spaces see, for example, Aliprantis and Border (1999)) and the space of profiles of strategies is metric, the existence of essential games is guaranteed by conditions which allow the map F to be upper semicontinuous. The focus of this note is just to show under which conditions, remarkable in
the framework of strategic choices and weaker than continuity of payoffs, the map \( F \) is upper semicontinuous with non-empty and compact values.

2. Preliminaries.

In order to obtain that the set-valued function \( F \) satisfies the hypothesis of Theorem 1, by using conditions over the payoffs of games weaker than continuity, let us remind a recent generalization of the continuity of real valued function: see Morgan and Scalzo (2007).

**Definition 1.** Let \( f \) be a real valued function defined on a topological space \( X \) and let \( x_0 \in X \). We say that the function \( f \) is upper pseudocontinuous at \( x_0 \) if

\[
\limsup_{x \to x_0} f(x) < f(x_0)
\]

for any \( x \in X \) such that \( f(x_0) < f(x_1) \). The function \( f \) is said to be lower pseudocontinuous at \( x_0 \) if

\[
\liminf_{x \to x_0} f(x) < f(x_0)
\]

for any \( x \in X \) such that \( f(x_1) < f(x_0) \). Finally, \( f \) is said to be pseudocontinuous at \( x_0 \) if it is both upper and lower pseudocontinuous.

The class of pseudocontinuous functions play a role in Choice Theory. In fact, as shown in Morgan and Scalzo (2007), if a continuous preference relation defined on a topological space – that is a complete and transitive binary relation such that the upper level sets and the lower level sets are closed – is endowed of numerical representations (also said utility functions), then all such representations are pseudocontinuous functions. So, pseudocontinuity is the common topological property among the numerical representations of continuous preference relations.

When a game \( y=(S_1, \ldots, S_m, f_1, \ldots, f_n) \) has the sets \( S_i \) compact and convex and the functions \( f_i \) pseudocontinuous on \( S_1 \times \cdots \times S_n \) and such that \( f_i(\cdot, x_{-i}) \) is quasi-concave for any \( x_{-i} \) and any \( i \), then, in light of Theorem 3.2 in Morgan and Scalzo (2007), \( y \) admits Nash equilibria. So, from now on, let \( Y \) be the set of all games \( y=(S_1, \ldots, S_m, f_1, \ldots, f_n) \) having the sets of strategies \( S_1, \ldots, S_m \) non-empty, compact, convex and included, respectively, in the subsets \( X_1, \ldots, X_n \) of normed spaces, and the payoff functions \( f_i \) are pseudocontinuous and bounded on \( X_1 \times \cdots \times X_n \) and such that \( f_i(\cdot, x_{-i}) \) is quasi-concave for any \( x_{-i} \). Now, we introduce a suitable topology on \( Y \): following Yu (1999), we first consider the metric on the space of all vector functions \( f = (f_1, \ldots, f_n) \), with \( f_1, \ldots, f_n \) pseudocontinuous and bounded on \( X \), defined as follows:

\[
\rho(f, f') = \sum_{i=1}^n \sup_{x \in X} |f_i(x) - f'_i(x)|,
\]

then, if \( K_i \) denote the set of all non-empty, compact and convex subset of \( X_i \), for any \( i \), we take the Vietoris’ topology (see Klein and Thopson (1984)) on \( K_i \) and so the topology \( \sigma \) on \( K = K_1 \times \cdots \times K_n \) which is the product of the Vietoris’ topologies on any \( K_i \). Finally, we obtain a topology \( \tau \) on \( Y \) as the product of \( \sigma \) and the topology induced by the metric \( \rho \).

3. The results.

Let \( F : Y \to 2^X \) be the set-valued function such that \( F(y) \) is the set of all Nash equilibria of the game \( y \), where \( Y \) is defined as in the previous section. We have the following result:
Theorem 2. The set-valued function $F$ has non-empty and compact values and is upper semicontinuous with respect to the topology $\tau$, that is: if an open set $O$ contains the set of Nash equilibria $F(y)$ of a game $y$ and if $(y^\alpha)_\alpha$ is a net of games converging to $y$ in the topology $\tau$, then we have $F(y^\alpha) \subseteq O$ for any game $y^\alpha$ with $\alpha \succ \alpha_0$ for a suitable index $\alpha_0$.

The proof of Theorem 2 can be achieved by the arguments of the proof of Theorem 3.2 in Scalzo (2008). Let us remark that a previous result on the upper semicontinuity of the set-valued function $F$ is given in Yu (1999) in the case in which the payoffs of any game are continuous functions. Here, we not only generalize the previous result, but we also obtain a result by using an ordinal topological property, that is the pseudocontinuity. Theorem 2 can be used in order to state that there exist essential games with pseudocontinuous payoff functions. In fact, if we recognize a non-empty subset $Y_1$ of $Y$ such that $Y_1$ is a Baire space, from Theorem 2 we know that $F$ is upper semicontinuous on $Y_1$ and from Theorem 1 we know that there exists a subset $Z \subseteq Y_1$ which is dense – that is: the topological closure of $Z$ coincides with $Y_1$ – and such that the map $F$ is also lower semicontinuous at $z$ for any $z \in Z$, which means that any game $z \in Z$ is essential.

Obviously, the pseudocontinuity is not the only generalization of ordinal character of the continuity in the setting of non-cooperative games. An other remarkable class of discontinuous games is the class of better-reply secure games, due to Reny (1999): a game $y=(S_1,\ldots,S_n, f_1,\ldots,f_n)$ is said to be better-reply secure if for any pair $(x^*, u^*)$ such that $x^*$ is not a Nash equilibrium of $y$ and $u^*$ belongs to the closure of the graph of the vector function $(f_1,\ldots,f_n)$, then some player $i$ has a strategy $x_i^*$ such that $f_i(x_i^*, x_{-i}) > u_i^* + \varepsilon$ for all $x_{-i}$ belonging in some neighbourhood of $x_{-i}^*$, where $\varepsilon$ is a suitable positive real number. We remark that any game with pseudocontinuous payoff functions is also a better-reply secure: see Proposition 4.1 in Morgan and Scalzo (2007).

Hence a question arises: Does the thesis of Theorem 2 hold for the class of better-reply secure games? The answer of the question is in the following theorem:

Theorem 3. Let $Y_1$ be the set of all better-reply secure games with spaces of strategies included, respectively, in $X_1,\ldots,X_n$, and let $F_1 : Y_1 \rightarrow 2^X$ be the set-valued function such that $F(y)$ is the set of Nash equilibria of $y$ – in light of Theorem 3.1 in Reny (1999), $F(y)$ is non-empty and compact. Then, there exist a game $y \in Y_1$, an open set $O$ containing $F_1(y)$ and a net of games $(y^\alpha)_\alpha$ converging to $y$ such that $F_1(y^\alpha) \setminus O$ is non-empty for any $\alpha$ - in other words, the map $F_1$ is not upper semicontinuous at $y$.

For a sketch of proof, it is sufficient to consider the game $G = (S_1, S_2, f_1, f_2)$ and the sequence of games $G^\alpha = (S_1^\alpha, S_2^\alpha, f_1^\alpha, f_2^\alpha)$ such that: $S_1 = S_2 = [0,1]$, $S_1^\alpha = S_2^\alpha = \left[0,1+\frac{1}{n}\right]$, $f_1(x_1, x_2) = f_1^\alpha(x_1, x_2) = h(x_1)$ and $f_2(x_1, x_2) = f_2^\alpha(x_1, x_2) = h(x_2)$, where the function $h$ is defined as follows:

\[
\begin{align*}
h(0) &= 1 \\
h(x) &= 0 \forall x \in \left[0,1\right] \\
h(x) &= x \forall x \in \left[1,\infty\right]
\end{align*}
\]
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