Robust Portfolio Management

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Abstract: *We define and compare robust and non-robust versions of Vol-VaR- and CVaR- portfolio selection models showing that robust CVaR is coherent, easy implementable and the most efficient.*

Keywords: Portfolio Selection, Robust optimization, Value at Risk, Conditional Value at Risk.

1.Introduction

Classical Portfolio Selection models are based on a bi-criteria optimization scheme whose goal is to form a portfolio in which expected return is maximized while some index of risk is minimized. The leading example is the famous and Nobel-awarded Markowitz's model (1991). Recently, however, two relevant problems have been observed: 1) positive and negative deviations of portfolio returns from their means play an asymmetric role in investor's utility; 2) the model's parameters (moments) are not known with certainty and optimal solutions depend heavily on their levels and perturbations.
As to the first point, financial practice and related theory has shown increasing interest towards downside risk and quantile based measures, such as Value at Risk (VaR). However, VaR if studied in the framework of coherent risk measures (Artzner *et al,* 1997), lacks reasonable properties like sub-additivity and convexity, in the case of general loss distributions.
This drawback entails both inconsistency with the well accepted principle of diversification as well as difficulties from the point of view of numerical tractability. To overcome these problems, recent literature on Portfolio Selection focused on coherent risk measures and in particular Conditional Value at Risk (CVaR) that is considered as the right objective to be minimized.
Rockafellar and Uryasev (2000 and 2002) and also Pflug (2000) proved that CVaR is a coherent risk measure in general, taking into account the 'tail risk'. Furthermore its formulation makes it possible to minimize CVaR using the methods of Linear Programming (LP).
Another interesting aspect is that by solving a simple convex optimization problem in one dimension it is possible to obtain simultaneously $CVaR_\alpha$ and VaR_α of the portfolio $\mathbf{x}$. This result is particularly important because it allows us to calculate the CVaR of the financial position without knowing in advance the value of VaR.
The second weakness point of classical portfolio selection models is the unpleasant property that optimal solutions depend heavily on parameter perturbations. In recent years, this feature has been dealt with through a new methodology introduced in the optimization literature under the name of Robust Optimization (Ben Tal and Nemirovski, 2002, Goldfarb and Iyengar, 2003). In the following, we adopt this approach by applying robust optimization to both symmetric and asymmetric measures of risk, providing, therefore, a joint solution to both mentioned problems of portfolio selection. Note that, in general, the robust reformulation needs not to be linear even in situation in which the original problem is linear. An important feature of our analysis is that we are able to avoid this problem and obtain a linear (for the minimization of VaR and CVaR) and a quadratic (in the case of Vol model) robust reformulation of our problems for which standard

minimization procedures are available. This goal is reached through the use of Soyster's (1973) approach to Robust Optimization.
In order to evaluate and compare the advantages related with the use of the robust counterpart of the three models of portfolio selection, we performed an implementation of the models both in a robust and a non-robust way. The comparison is done through an ex-post analysis on the results obtained by the ex-ante implementation of each model in selecting from a set of 28 European hedge funds during 2007, a "wonderful" year of data for model stressing and backtesting. As we shall see, the strategies obtained by means of the robust approach have a definitely better performance and, among the robust models, CVaR dominates the other competitors because of its coherent nature.

2. Robust Optimization

Deterministic (non robust) optimization problems are formulated assuming that the parameter values are known. Problems of risk management with uncertain parameters and minimization of VaR or CVaR are known as Stochastic Programming (SP) problems (Zenios, 1993 and Rockafellar and Uryasev, 2002) and every SP model can be viewed as an extension of a deterministic model where the uncertain parameters have been given a probabilistic representation.
However it was noted (Ben Tal and Nemirovski, 1999 and 2002) that generally, using a SP approach, constrains can be violated with certain probability and, as a consequence, in a model of SP with data uncertainty, the variables do not necessarily satisfy the original constraints, but only a relaxation of them (soft constraints).
Robust Optimization (OP) can overcome the drawback of SP The goal is to find a (robust) solution which is feasible for all possible data realizations and optimal in some respect. The first attempt to formalize the idea of uncertain hard constraints in LP models is due to Soyster (Soyster, 1973). According to this approach, the constraints of the robust counterpart of the original uncertain problem are completely equivalent to a system of linear constraints and therefore the robust counterpart becomes a convex problem and it can be easily and quickly solved.
It must be noted that this approach is extremely conservative because the constraints of the robust counterpart correspond to the case when every entry in the constraint matrix is as "bad" as it could be, while in real situations, the coefficients of the constraints are not simultaneously at their worst values. Moreover, such an approach is able to reach the optimum in a few seconds with standard processors and high dimensional problems.
In this paper we want to implement both in a robust and non-robust way the minimization models of VaR, CVaR and variance. The uncertainty is in the parameter vector $\mathbf{r}$ containing the expected returns of each asset so that we obtain the following general robust model:

$$\min_{\mathbf{x}} \{f(\mathbf{x}),\ \min_{\mathbf{r}\in I} \{ \mathbf{r}^{T} \mathbf{x} \geq g\}, \mathbf{1}^{T} \mathbf{x} = 1 \} \tag{1}$$

where $\mathbf{x}$ is the vector of portfolio weights, f($\mathbf{x}$) represents the different risk measures, g is a scalar that denotes the minimum average return that is required by the optimal portfolio and the constraint $\mathbf{r} \in I$ denotes that the expected returns of the assets are not point values but may vary within their ranges of variability described by the uncertainty set I. Now, following Soyster's approach, stating that the parameters of the vector $\mathbf{r}$ are known within particular ranges that are the confidence regions of the parameters' estimates, it is possible to define the uncertainty set as $I=\{\mathbf{r} \mid r'_i - s_i \leq r_i \leq r'_i + s_i,$ for all $i = 1,\dots, N\}$. As a consequence, the semi-infinite robust counterparts of the considered models will be computationally tractable because it is possible to show that, in this case, we have (Goldfarb and Iyengar, 2003):

$$\min_{\mathbf{r}\in I} \mathbf{r}^{T} \mathbf{x} = \mathbf{r}'^{T} \mathbf{x} - \mathbf{s}^{T} |\mathbf{x}| \tag{2}$$

Using a new variable **m** to replace |**x**| by the constraints $m_i \geq x_i$ and $m_i \leq x_i$, we obtain the following robust model:

$$\min_{\mathbf{x}} \{f(\mathbf{x}),\ \mathbf{r}'^{T}\mathbf{x} - \mathbf{s}^{T}\mathbf{m} \geq g,\ m_i \geq x_i,\ m_i \leq x_i,\ \mathbf{1}^{T}\mathbf{x} = 1,\ i=1,\ldots,N\} \tag{3}$$

where i denotes the assets, r'_i is the estimated value of the expected return of the i-th asset, s'_i is the standard deviation of this estimate. These models can be easily solved at low computational costs.

3. Empirical results

In this section we will show how the methodology of Robust Optimization can be implemented to minimize the VaR, the CVaR and the Volatility of a portfolio of hedge funds to find an optimal strategy of asset selection and dynamic rebalancing. In particular we will implement the models described at the end of section 2 both in a robust and non-robust way with the aim to operate some comparisons that could help to evaluate the relative advantages of these methodologies of asset management.
We will assume that uncertainty is referred only to the vector of the expected return of the considered assets. VaR and CVaR are obtained by an appropriate historical simulation, while the variance in the Markowitz's model is obtained by the usual historical estimation procedure.
In the empirical implementation we have considered the time series from January 9th, 2007 to September 18th, 2007[1] of the weekly prices of 28 "Absolute Return" funds quoted in euro from Bloomberg[2].
Using GAMS[3] we have written a program for the minimization of the risk measures using as minimum threshold (g) of weekly portfolio returns the value at the beginning of every week of the 3-month Euribor rate as the closest approximation of a riskless asset in the market. The confidence level chosen for the quantification of the VaR and CVaR has been fixed at 99%. Assuming that the knowledge of a large number of historical data gives an exhaustive set of scenarios and guarantees the reliability of risk measures (VaR and CVaR) obtained by historical simulation, the decisive factor is the estimation of the expected returns of the Funds.
Because of their random nature and their crucial role in the selection model, special attention has to be paid to an efficient evaluation of the parameters r'_i. They are estimated weekly for all the weeks in the period January 9th, 2007 to September 18th, 2007 (with an increasing window starting at April 4th, 2006) as one step-ahead linear predictors of the following ARIMAX(1,0,0) with GARCH(1,1) regression errors:

$$r_i(t) = a + b\, r_i(t-1) + c_1 r_B(t-1) + c_2 r_B^2(t-1) + c_3 r_B^3(t-1) + d_1 r_S(t-1) + d_2 r_S^2(t-1) + d_3 r_S^3(t-1) + u_i(t)$$
$$u_i(t) \approx IN(0, h_i(t))$$
$$h_i(t) = v_i + p\, h_i(t-1) + q\, u_i^2(t-1) \tag{4}$$

where r_B is the weekly rate of return of the JP Morgan Emu Government Bond index and r_S is the weekly return of the MSCI Emu Stock market index.
The lower and upper limits $r'_i \pm s'_i$ are obtained as a 60% confidence interval. It may be interesting to note that the true fund returns at *(t+1)* always lie within the confidence intervals estimated at time *t*,

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[1] Note that this period of time has been characterized by a first stock market shock at the end of February and a disrupting crash and deep market crisis starting from July (subprime mortgages meltdown) and it is therefore an ideal time period to test "absolute return" strategies, aimed at reaching an absolute alpha goal with minimum risk and low correlation with bond and equity markets.
[2] In particular the 28 Funds are Absolute Return hedge funds managed by CAF, Fortis, Invesco, Julius Baer, JP Morgan, Pictet, Schroeder and UBS.
[3] General Algebraic Modelling System, GAMS--IDE GAMS Rev. 135 Vis. 21.1 135.

a circumstance particularly important for the construction and implementation of the robust version of the portfolio selection models.
The results of our comparisons show that:
i) in about 9 months, the risk-free roll-over portfolio reached a 2.97% rate of return, beating all non-robust models. Viceversa, it was left behind by all robust models, whose returns were in excess of 7%; CVaR obtained the maximum return (7.89%);
ii) the ex-post volatility of robust models is half time as high as the vol of their non-robust counterparts;
iii) all robust models have the higher (positive) skewness against the negative skewness of non-robust portfolios. The skewness of robust CVaR and Vol models are the higher. The kurtosis of CVaR and VaR models are the lower;
iv) the ex-ante CVaR of the optimal CVaR-portfolios is uniformly the minimum risk level attained by all models;
v) the truly robustness of the models has been particularly evident during the subprime crisis in the second half of 2007. Whilst non-robust portfolios were heavily cut down by large losses in many weeks, robust strategies were sheltered from the crisis and passed through without significant damage; CVaR model was the best performer during the market storm;
vi) using the robust CVaR model there would never be any loss, while losses can occur in the robust VaR and Vol models as well as in all non-robust approaches; moreover, the realized returns of robust strategies are almost always larger than the threshold value g;
vii) the ex-post analysis allows us to be confident in the evaluation of the portfolio riskiness because we never obtained losses greater than the forecasts implied by the associated risk measures.
Then, using 2007 historical data, we find that robust models overcome their non-robust, traditional counterparts and, in particular, the robust CVaR model, in a period of significant credit, bond and equity shocks, provides optimal portfolios with minimum risk, high skewness, low kurtosis and high returns, above the risk-free portfolio and the other competing risk models.

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