Computational Workout: Division Tables as Training Exercises*

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Abstract

This paper examines *ostraka* O.Petr.Mus. 64 and 65 (TM 65801 and TM 113882; second half of the 5th cent., Tentyris?), inscribed with division tables for 31 and 57, respectively. It argues that the tables did not serve as ready reckoners and were not copied after a model text, but rather recorded the results of actual computations performed for the purpose of training numerical skills. To facilitate the discussion, the paper first provides a short introduction to the computation of fractional quotients.

Keywords

Numeracy, division tables, education

A large part of the papyrological evidence for scribal, as opposed to academic, mathematics from Greco-Roman Egypt consists of numerical tables for basic arithmetical operations of addition, multiplication, and division.[1] About one hundred specimen are presently known, with division tables outnumbering those for addition and multiplication.[2] The division tables inscribed on two *ostraka*, O.Petr.Mus. 64 and 65 (TM 65801 and TM 113882; second half of the 5th cent. A.D., Tentyris?), are remarkable in that they present otherwise unattested tables for the divisors 31 and 57, with 31 also being the highest prime divisor to feature in a division table in the papyrological record. Since divisions by these divisors were not likely to be of much practical use, the *ostraka* raise the question of how and for what purpose the tables were produced. Before addressing this question more closely,

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[1] For the distinction between scribal and academic mathematics, cf. Jones 2009, 339-343. For an overview of numerical tables, cf. also Fowler 1999, 234-240. Less attested types of arithmetical tables include tables of squares (cf. Fowler 1999, 239-240); for a table of powers of 2, cf. P.Cairo S.R. 3069 v (TM 703093; 2nd cent. B.C., Minia?), published in Aish 2016, 49-54.

[2] Fowler 1999, 238, 269-275; Azzarello 2018, 95, estimates that division tables account for about 2/3 of all published arithmetical tables; cf. also Jones 2009, 340-341.

however, a few words on fractional notations and division tables in papyrological evidence are in order.

The most peculiar aspect of ancient division was the notion of the fractional quotient, i.e. of a division $m \div n$ where $n>m$ and $m>1$, which was conceived not as a common fraction, but as a sum of a series of distinct unit fractions (i.e. where the numerator was 1), except that a notation for two-thirds was usually admissible. To obtain a quotient of 7 divided by 10, for example, one had to find a series of unit fractions the sum of which would amount to the value of 7÷10. Possible solutions could be $\frac{1}{2} + \frac{1}{5}$ or $\frac{2}{3} + \frac{1}{30}$, which would be recorded as ∠ ε′ or ω λ′, respectively, where ∠ was a symbol for one-half, ω for two-thirds, and ε′ and λ′ for one-fifth and one-thirtieth, with the little tick (′) by the numeral five (ε) or thirty (λ) indicating that it was not a natural number but a fraction, or a part.[3]

Division tables were arranged by divisors, each series listing quotients of a division with the same divisor and consecutively increasing dividends (from 1 to 10,000 for the divisors up to 10; up to the value of the divisor thereafter).[4] Modern editions make use of the ancient notation to indicate fractions by little ticks, that is $1/n = n'$ (with 3″ used for 2/3), and employ it when referring to a division table by its divisor as a table of *n*′, that is, of *n*th parts.

Individual division tables could be compiled into larger sets, usually in the order of increasing divisors. Thus, the codex P.Cair. cat. 10758 contains a comprehensive sequence of division tables for divisors from 3′ through 20′, as well as a series for two-thirds (3″), while a partially preserved roll SB XX 15071 + P.Mich. III 146 has tables for 7′ through 18′. Sets comprising tables for selected divisors, however, are more common, whereas wooden tablets and especially *ostraka* often contain a table for a single divisor.[5] Overall distribution of preserved tables in respect to their divisors is uneven: Those for divisors below 10 are most numerous, almost twice outnumbering those for divisors between 11 and 20, whereas very few feature divisors above 20.[6]

[3] Both expressions of the quotient of 7÷10 are attested in papyrological evidence. The first can be found in the roll SB XX 15071 + P.Mich. III 146, line 129 (TM 64346; 3rd-early 4th cent. A.D., Fayum?), for which see now Azzarello 2019, and in P.Cair.cat. 10758 (often referred to as the Akhmim Mathematical Papyrus), edited in Baillet 1892, fol. 1 v, col. 8 (TM 64999; late 4th or 5th cent. A.D., Panopolis?). For the date of this codex, see now Bagnall / Jones 2019, 3 n. 8. The second decomposition appears in a yet unedited part of a wooden-tablet codex from the Sayce Collection in the Ashmolean Museum (TM 61276; 3rd cent. A.D., Thebes?), for the description of which see Parsons 1970, esp. 142-143; it is also recorded in P.Rain.UnterrichtKopt. 332, fol. 7 v, col. 1, line 11 (9th cent. A.D., provenance unknown).

[4] For detailed description and analysis of the formats of division tables, including the composition of the header that often featured the division of 6000, cf. Azzarello 2018, esp. 95-97.

[5] For composition of the tables, see the catalogue of division tables in Fowler 1999, 269-274. Examples of single series on *ostraka* include, besides O.Petr.Mus. 64 and 65, tables for 7′ on O.Sarga 24 (TM 89510), 25 (TM 89511), and 27 (TM 89513, with Lougovaya 2020), and for 11′ on O.Sarga 26 (TM 89512), all dated to 5th-7th cent. A.D.; a table for 2/3 is preserved on O.Mich. inv. 9733 (TM 64127; 3rd cent. A.D., Soknopaiou Nesos), published in Youtie 1975.

[6] See the table summarizing the data for 172 division tables (i.e. individual division series) in Fowler 1999, 238. Note that the record for tables for 25′ and 49′ (cf. Fowler 1999, 270, no. 11) should be now deleted, because the *ostrakon* believed to contain them, O.Sarga 27, has been shown to have only a table for 7′, see Lougovaya 2020.

While it is common sense that the high number of division tables must be the consequence of their frequent use, presumably to help learn or perform divisions, it is far from clear how exactly they served this purpose or how particular exemplars were produced. Possible options include copying from a model or from dictation, reproduction from memory, actual computation of the divisions, or some combination of these procedures. The prevailing scholarly opinion seems to be that the tables preserved in papyrological evidence were normally produced by copying from existing models, so much so that it is possible to speak not only of the continuity of a computational tradition stretching over several centuries, but also of a textual tradition transmitting the same decompositions and of deviating quotients as variants.[7]

As for usage, it is generally assumed that the tables served as a learning aid in school where they were memorized, while more extensive compilations were ready reckoners for those charged with various kinds of accounting.[8] Curiously, however, a quick look at the data in the Papyrological Navigator suggests that the absolute majority of fractional quantities recorded in papyrological documents such as accounts or contracts come from divisions by 2, 3, and their multiples (most common being the progression 1/2 1/4 1/8 1/16 1/32 etc., but also 1/3 1/6 1/9 1/12 1/24 1/48 etc.), followed by those resulting from divisions by 5 and its multiples.[9] Divisions by 11, 13, 17, or 19 in general and especially fractional quotients beyond 1/*n* that would have resulted from such divisions are rare to non-existent.[10] Although more analysis of computational practices employed in documentary accounts is needed, it does not look that, let's say, tax-collectors would have had much need to consult division tables for 11, 13, 17, or 19. The fact that quite a few examples of such tables survive is likely to point to their use in an educational environment.

Returning to the O.Petr.Mus. *ostraka* with their tables for 31′ and 57′, we can probably rule out their use as ready reckoners. Should a need to divide by 31 or 57 ever arise in a "real-life" situation, one imagines the ancients would have resorted to an approximation and divided by 32 or 60, which would entail incomparably easier computations and no greater margin of error than one routinely finds in approximated calculations of areas of land, for example. The question that presents itself then is for what purpose the tables for these uncommon divisors were composed, and, what is related, how the quotients recorded in them were produced. The clues to the answers to these questions lie precisely in the cumbersomeness of dividing by 31.

[7] See, for example, Knorr 1982, 147-151; Azzarello 2108, 2019. Although Fowler (1999, 237) notes that «the tables must have been frequently recomputed, when occasion demanded», he does not elaborate.
[8] Fowler (1999, 235) writes that quotients preserved in papyrological evidence «would have been either memorised or consulted». Writing about P. Mich. inv. 621 (= P.Mich, III 146, cf. fn. 3 above), Karpinski (1923, 24) declares that «[u]ndoubtedly these tables were used in the offices of tax collectors where it was necessary to compute fractional parts of money». Parsons (1970, 142) concurs: «no doubt they served as ready reckoners».
[9] I am grateful to James Cowey for helping me retrieve the data from the PN. My observations at this point, however, are preliminary and a careful analysis of the data remains a desideratum.
[10] For the rare use of 1/11 for areas of land, cf. Nielsen 1992, 150.

Not all divisors are created equal. Performing divisions with common divisors such as 2, 3, 5, and their low multiples entails relatively simple computations resulting in straightforward decompositions. If a division table for such a divisor contains no mistakes it is difficult if not impossible to ascertain how it was produced. Divisions with larger divisors, however, especially with prime numbers beyond a single-digit, tend to be more challenging and the ways of computing them less evident. Because of that, quotients recorded for such divisions can be indicative of the process by which they were arrived at. To understand better how a result could be influenced by the way in which a fractional quotient was computed, let us perform a couple of divisions for a two-digit prime divisor.[11]

Suppose we want to divide 4 by 29.[12] Since 29 is not divisible by 4, the quotient will consist of several fractions, the first of which, we estimate, has to be smaller than $\frac{1}{7}$, that is, has to have a denominator larger than 7 (because $\frac{1}{7} = \frac{4}{28}$, and $\frac{4}{28} > \frac{4}{29}$). Thus, the largest the first fraction of the quotient can be is $\frac{1}{8}$. To peel $\frac{1}{8}$ off $\frac{4}{29}$ we need first to scale up the division 4÷29 by 8, that is, to multiply by 8 both the dividend and the divisor (or, the numerator and denominator if the division is rendered as a common fraction), which gives 32÷232. Since 32 can be decomposed, that is broken down, into factors of 232, 32 = 29 + 2 + 1, the scaled up division can be expressed as a sum of three fractions, (29 + 2 + 1) ÷ 232, the conversion of which to lowest terms will result in a sum of unit fractions, 8′ 116′ 232′. The process can be recorded in modern notation as follows:

$$\frac{4}{29} = \frac{4\times 8}{29\times 8} = \frac{32}{232} = \frac{29+2+1}{232} = \frac{29}{232} + \frac{2}{232} + \frac{1}{232} = \frac{1}{8} + \frac{1}{116} + \frac{1}{232}$$

Since there exist various (in fact an infinite) number of ways in which a division of *m* by *n* can be decomposed, the choice of a particular decomposition may reflect a preference for a certain kind of quotient. The decomposition we just computed is the one with the largest leading fraction, a preference that can be imposed as a condition, i.e. «compute the division of 4 by 29 so that the first fraction of the quotient is the largest possible».[13]

Another preference could be to avoid very small fractions in the quotient. This can also be set as a condition, for example: «compute the division of 4 by 29 so that no denominator in the quotient is larger than 200». Should this be the case, the quotient we computed above would not satisfy the condition, and we would have to find a different decomposition. There is no algorithm for how to do

[11] In performing division I am following the methods described in the solutions to problems with fractions in P.Cair. cat. 10758, as well as the solutions conjectured for partition problems in Bagnall / Jones 2019, 52-53.

[12] No example of the divisions of 4 and 5 by 29 performed here survives in papyrological evidence and thus the computations are conjectural, but P.Mich. III 145 (TM 63556; 2nd cent. A.D., provenance unknown) contains a fragmentary division table for 29 with the entries for the dividends from 12 through 17.

[13] In the partially preserved division tables for 23 and 29 in P.Mich. III 145, there seems to be a preference for maximizing the first or the first two fractions in the decompositions, cf. Knorr 1982, 142.

it and the process is exploratory.[14] We may realize that, if we scale up the division not by 8, but by 12, we obtain 4÷29 = 48÷348, and that 48 can be decomposed into factors of 348, because 48 = 29 + 12 + 4 + 3. Consequently, 48÷348 can be partitioned into four fractions, the conversion of which to lowest terms results in a sum of unit fractions, 12′ 29′ 87′ 116′, in modern notation:

$$\frac{4}{29} = \frac{4 \times 12}{29 \times 12} = \frac{48}{348} = \frac{29 + 12 + 4 + 3}{348} = \frac{29}{348} + \frac{12}{348} + \frac{4}{348} + \frac{3}{348} = \frac{1}{12} + \frac{1}{29} + \frac{1}{87} + \frac{1}{116}$$

Suppose now that our computation of 4 divided by 29 was an entry in a table. Since a division table is nothing more than a series of division problems in which the divisor *n* remains the same, but the dividend increases consecutively, we now need to find the quotient of 5 divided by 29. To do so, we can simply add one twenty-ninth to the quotient of 4÷29 (it helps to think of division tables as tables of parts: if 4÷29 is 4 of one twenty-ninth parts, then 5÷29 would be 5 of one twenty-ninth parts). We would take the first quotient we computed above, 4÷29 = 8′ 116′ 232′, and by adding 29′ to it would arrive without any computation at 8′ 29′ 116′ 232′ as the quotient of 5÷29. This is attractively easy; however, had we computed the quotient anew, we could have produced a much more elegant solution, 5÷29 = 6′ 174′, which has a larger leading fraction, a smaller number of fractions, and a smaller last denominator than the quotient we found without calculations.

Our exercise allows us to draw two conclusions: (a) the quotient of a division may reflect a condition imposed on it, e.g. a preference for the largest first fraction or a limit on the value of the smallest fraction; (b) in the case of consecutive quotients as in a division table, it may be possible to determine the method by which they were obtained, i.e., with our without computing. With this in mind, let us now look at O.Petr.Mus. 64, which preserves the table for the next prime number (31) after the one we experimented with. In what follows I first give the text of the table followed by the analysis of quotients with conjectural step-by-step computations. I then demonstrate that some properties of the quotients can be best accounted for if the table was produced as an exercise meant to train the numerical skills of its «computer» – as I call the person who performed the computations – and that the workout was purposefully made more challenging through the imposition of certain stipulations on the quotients. The table for 57, a multiple of a two-digit prime 19, on O.Petr.Mus. 65 was probably produced by the same person and as a result of a similar task. The text follows that of Giuseppina Azzarello in O.Petr.Mus. 64 and 65.[15]

[14] Cf. the discussion of essentially the same kind of computations in Bagnall / Jones 2019, 50-52.

[15] Cf. Azzarello 2008, 159-167, for detailed notes on the readings and the editorial history of the piece; the text published in O.Petr.Mus. 64 takes into account a small fragment known to Crum but subsequently lost, until its rediscovery in the Museum after the publication of Azzarello 2008. The text here differs from the edition in that (a) the fraction tick (′), which does not appear on the *ostrakon*, is not added to the numbers indicating fractions in the Greek text, but is used in the translation to record *n*′; (b) the occasionally used interpunct is moved from the apparatus into the Greek text; (c) a different restoration is adduced for the quotient in line 18.

Recto

τὸ λ̣[α ἀριθ(μῶν)?] ρϟγ ∠· λα·	the 3[1st of 6000] 193 2′ 31′	
ξβ· τ̣[ῆς μῖας (?) τ]ὸ λα λα //	62′ of [one], the 31st is 31′ //	
τῶν β λ̣[α] ξ̣β ϟγ ρπϛ	of 2	3[1′] 62′ 93′ 186′
τῶν γ ιβ ρκ̣δ̣ ρπϛ·	of 3	12′ 124′ 186′
τῶν δ ιβ· λα· ρκδ· ρπϛ	of 4	12′ 31′ 124′ 186′
τῶν ε ιβ κ ϟγ ρ̣ν̣ε̣ ρπϛ	of 5	12′ 20′ {93′} 155′ 186′
τῶν ϛ ιβ· κ· λα ϟγ ρνε ρπϛ	of 6	12′ 20′ 31′ {93′} 155′ 186′
τῶν ζ ϛ· λα ξβ ϟγ	of 7	6′ 31′ 62′ 93′
τῶ̣ν η̣ ḍ ρ̣κδ	of 8	4′ 124′
[τῶν θ d λα ρκ]δ̣	[of 9	4′ 31′ 12]4′
[τῶν ι d κ ξβ ρν]ε̣	[of 10	4′ 20′ 62′ 15]5′
[τῶν ια γ ξβ ρπϛ]	[of 11	3′ 62′ 186′]
[τῶν ιβ γ λα ξβ ρ]π̣ϛ̣	[of 12	3′ 31′ 62′ 1]86′
[τῶν ιγ γ κ ξβ ρκ]δ̣ ρ̣ν̣ε̣ ρ̣π̣ϛ̣	[of 13	3′ 20′ 62′ 12]4′ 155′ 186′
[τῶν ιδ γ ιβ ξβ ϟγ] ρκδ̣	[of 14	3′ 12′ 62′ 93′] 124′
[τῶν ιε γ ιβ λα] ξ̣β̣ ϟγ ρκδ	[of 15	3′ 12′ 31′] 62′ 93′ 124′
[τῶν ιϛ ∠ ξβ]	[of 16	2′ 62′]

Verso

τῶν ιζ ∠· λα̣ [ξβ]	of 17	2′ 31′ [62′]
τῶν ιη ∠· κ [ξβ ρκδ ρ]ν̣ε	of 18	2′ 20′ [62′ 124′ 1]55′
τῶν ιθ ∠· ι̣β̣ [ξβ ρκδ ρ]πϛ	of 19	2′ 12′ [62′ 124′ 1]86′
τῶν κ ∠ ιβ [κ ρν]ε ρπϛ	of 20	2′ 12′ [20′ 15]5′ 186′
τῶν κα̣ ∠ ι̣β̣ κ̣ λα ρνε ρπ̣ϛ̣	of 21	2′ 12′ 20′ 31′ 155′ 186′
τῶν κβ ϖ λ̣α̣ ϟ̣γ	of 22	3″ 31′ 93′
τῶν κγ ϖ κ ϟ̣γ̣ ρκδ ρνε	of 23	3″ 20′ 93′ 124′ 155′
τῶν κδ ∠ d ξβ ρκδ	of 24	2′ 4′ 62′ 124′
τῶν κε ∠ d λα̣ ξβ ρκδ	of 25	2′ 4′ 31′ 62′ 124′
τῶν κϛ ∠̣ γ· ρπϛ	of 26	2′ 3′ 186′
τῶν κζ ∠ γ λα̣ ρ̣π̣ϛ̣	of 27	2′ 3′ 31′ 186′
τῶν κη ∠ γ· κ̣ [ρκδ ρνε ρπϛ]	of 28	2′ 3′ 20′ [124′ 155′ 186′]
τῶν κθ̣ ∠ γ ι̣[β ϟγ ρκδ]	of 29	2′ 3′ 1[2′ 93′ 124′]
τῶν λ ∠ γ ι̣[β λα ϟγ ρκδ]	of 30	2′ 3′ 1[2′ 31′ 93′ 124′]
τῶν λα α̣	of 31	1

2 λα// ex δα// 18 τῶν ιζ ∠ · λα̣ [ϟγ ρπϛ] Azzarello

Notes on Computations

Lines 1-2. First, the writer needed to find the quotient of 6000 by 31. Its integer part is 193, remainder 17. To divide 17 by 31, he had to scale up the division 17÷31 by 2 and then decompose 34 parts of 62′ into factors of 62, which are 31, 2 and 1, or, in modern notation:

$$\frac{17}{31} = \frac{17\times 2}{31\times 2} = \frac{34}{62} = \frac{31+2+1}{62} = \frac{31}{62} + \frac{2}{62} + \frac{1}{62} = \frac{1}{2} + \frac{1}{31} + \frac{1}{62}$$

Line 3. To find the quotient of 2÷31, recorded as 31′ 62′ 93′ 186′, our computer seems to have applied the algorithm $\frac{2}{n} = \frac{1}{n} + \frac{1}{2n} + \frac{1}{3n} + \frac{1}{6n}$, which is based on the fact that $1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{6}$, and with the help of which any 2/*n* can be decomposed into four unit fractions. The algorithm was usually avoided in division tables and exercises, either because it produces a series of fractions that was not viewed as optimal or perhaps also because it defied the purpose of exercise.[16] Computing 2÷31 with scaling up by 20 would have given 20′ 124′ 155′.

Lines 4-5. Since the quotient of 2÷31 contained 31′, the next entry had to be computed: scaling up by 12 produced 36 parts of 372′, which can be partitioned into 31 + 3 + 2 parts of 372′ and then converted to lowest terms, which are unit fractions 12′ 124′ 186′.

The quotient for 4÷31, was obtained by adding 31′ to the quotient of 3÷31, producing 12′ 31′ 124′ 186′.

Lines 6-7. The quotient of 5÷31 had to be computed, which turns out to entail somewhat more tedious calculations involving scaling up by 60.[17] This would result in 300 parts of 1860′. The correct decomposition of 300 into factors of 1860 is 155 + 93 + 30 + 12 + 10, which leads to the quotient 12′ 20′ 62′ 155′ 186′, in modern notation:

$$\frac{5}{31} = \frac{5\times 60}{31\times 60} = \frac{300}{1860} = \frac{155+93+30+12+10}{1860} = \frac{1}{12} + \frac{1}{20} + \frac{1}{62} + \frac{1}{155} + \frac{1}{186}$$

Our computer, however, made a mistake in decomposing 300 not as 155 + 93 + ***30*** + 12 + 10, but as 155 + 93 + ***20*** + 12 + 10, which led him to his quotient of 12′ 20′ ***93′*** 155′ 186′ (because he took 20 parts of 1860′, i.e. $\frac{20}{1860} = \frac{1}{93}$, instead of 30, i.e. $\frac{30}{1860} = \frac{1}{62}$). Unaware of the mistake, he extended it to the next entry, for 6÷31, which he produced by adding 31′ to the quotient of 5÷31, obtaining thereby 12′ 20′ 31′ 93′ 155′ 186′ in place of 12′ 20′ 31′ 62′ 155′ 186′. Had he computed the quotient,[18] he would have easily obtained the more elegant decomposition 6′ 62′ 93′.

Line 8. Since the previous (mistaken) quotient contained 31′, the quotient of 7÷31 had to be computed. This was done by scaling up the division by 6, decomposing 42 parts of 186′ into 31+ 6 + 3 + 2, and reducing the resulting fractions to lowest terms.

[16] Exceptionally, it was used to express 2/101 as 101′ 202′ 303′ 606′ in the so-called 2/*n* table in the Rhind Mathematical Papyrus. The table contains decompositions of 2/*n* for odd *n* from 3 to 101, and of the fifty recorded quotients only that one is obtained by application of the algorithm, which Knorr 1982, 138, calls «both obvious and disappointing». Scholarly literature on the Rhind Papyrus, an extensive mathematical papyrus containing problems and tables and written in the Second Intermediate Period after ca. 1550 B.C., is vast; for a brief description of the editorial history of the papyrus, see Imhausen 2016, 65-67, and for the editions, see Peet 1923; Chace / Bull / Manning / Archibald 1927-1929.

[17] Alternatively, the computations could have been done in several steps.

[18] This could have been done by scaling up 6÷31 by 6, peeling off 6′ and splitting the remaining 5 parts of 186′ into (2 + 3) ÷186.

Lines 9-10. The next two entries were easy: Scaling up by 4 suggests itself for computing 8÷31 since it produces 32 parts of 124′, which can then be split into 31 and 1 of 124′, corresponding to 4′ 124′. And the next entry is arrived at simply by adding 31′, i.e. 9÷31 = 4′ 31′ 124.

Lines 11-13. The quotient of 10÷31 had to be calculated. The division was scaled up by 20; the resulting 200 parts of 620′ were decomposed into 155 + 31 + 10 + 4, which, after conversion to lowest terms, produced 4′ 20′ 62′ 155′.[19] Although the computer could have added 31′ to get the quotient in the next entry, he chose to compute it, perhaps realizing that 11÷31 is larger than $\frac{1}{3}$ and thus that the quotient should begin with that fraction. He computed it as 3′ 62′ 186′. For the next quotient, he added 31′, producing 12 of 31′ = 3′ 31′ 62′ 186′.

Line 14. The quotient in this entry is significant. The easiest way to compute 13÷31 would be first by scaling up by 3 and peeling off 3′: $\frac{13}{31} = \frac{13 \times 3}{31 \times 3} = \frac{39}{93} = \frac{31+8}{93} = \frac{1}{3} + \frac{8}{93}$, and then decomposing 8 parts of 93′ into 12′ 372′, producing the quotient 13÷31= 3′ 12′ 372′. That our computer did not do it suggests that he had conditions imposed on the size of the denominator in the fractional part of the quotient. To meet them, he had to decompose 8÷93 by scaling it up by 20, which allowed him to arrive at the recorded quotient of 3′ 20′ 62′ 124′ 155′ 186′.

Lines 15-16. The quotient of 14÷31, recorded as 3′ 12′ 62′ 93′ 124′, was computed, even though the preceding entry did not feature 31′. One could imagine a stipulation whereby the fractional part could not contain more than six fractions, and thus the computer could not proceed simply by adding 31′. This is, however, what he did to produce the next entry, the quotient of 15÷31, which is given as 3′ 12′ 31′ 62′ 93′ 124′.

Lines 17-18. Scaling up by 2 is obvious for computing 16÷31 (= 2′ 62′), and for the next quotient one only needs to add 31′ (17÷31 = 2′ 31′ 62′). Although only the symbol for one-half and the fraction λα (that is, 31′) are visible on the *ostrakon*, the restoration ∠· λα̣ [ξβ] is to be preferred to ∠· λα̣ [ϟγ ρπϛ]. While both are, strictly speaking, correct, two considerations favor the former: (a) the writer is likely to have followed his standard procedure of adding, when possible, 31′ to the result of the previous quotient; (b) 2′ 31′ 62′ is recorded as the fractional part of the quotient of 6000 by 31, which corresponds to $\frac{17}{31}$ (6000 ÷31 = 193, Remainder 17).

Lines 19-22. The entry for 18÷31 had to be computed (= 2′ 20′ 62′ 124′ 155′), which was done by scaling up by 20. Although the computer could simply have added 31′ to get the quotient in the next entry, he instead chose to compute that and the following entry, obtaining 19÷31= 2′ 12′ 62′ 124′ 186′ and 20÷31= 2′ 12′ 20′ 155′ 186′, both of which are more compact decompositions than the addition of 31′ would have produced. Perhaps now tired of calculations, he resorted to adding 31′ to produce the next entry, 21÷31 = 2′ 12′ 20′ 31′ 155′ 186′. This was unfortunate: had he computed this division, he would have obtained the much more compact solution of 3″ 93′.

[19] The division may have been done in steps, with scaling first by 4 and peeling of 4′, then scaling up by 5 and peeling off 20′, and finally splitting the remaining 14÷620 into (10 + 4) ÷ 620, i.e. 62′ 155′.

Line 23. The quotient of 22÷31 (= 3″ 31′ 93′) was computed: the division was scaled up by 3, producing 66 parts of 93′, from which 3″ could be peeled off leaving 4 parts of 93′ to decompose, in modern notation: $\frac{22}{31} = \frac{22 \times 3}{31 \times 3} = \frac{66}{93} = \frac{62+4}{93} = \frac{2}{3} + \frac{3+1}{93} = \frac{2}{3} + \frac{1}{31} + \frac{1}{93}$.

At this point it should have become apparent that the quotient in the previous entry, for 22÷31, could have been expressed as 3″ 93′ instead of 2′ 12′ 20′ 31′ 155′ 186′.

Lines 24-26. Decompositions for 23÷31 (= 3″ 20′ 93′ 124′ 155′) and 24÷31 (= 2′ 4′ 62′ 124′) were computed; the first had to be because the preceding quotient featured 31′, whereas the reason for the second was possibly the realization on the part of our computer that the quotient could have 2′ 4′ as the leading fractions (since $\frac{24}{31} > \frac{24}{32}$ and $\frac{24}{32} = \frac{1}{2} + \frac{1}{4}$). The next entry, 25÷31 (= 2′ 4′ 31′ 62′ 124′), was produced by adding 31′ to the previous quotient. Had it been computed, it would likely have been 2′ 4′ 20′ 155′.

Lines 27-31. Three quotients were computed, 26÷31 (= 2′ 3′ 186′), 28÷31 (2′ 3′ 20′ 124′ 155′ 186′), and 29÷31 (=2′ 3′ 12′ 93′ 124′), and the other two were arrived at by the addition of 31′ to the decomposition in the previous entry, 27÷31 (= 2′ 3′ 31′ 186′) and 30÷31 (= 2′ 3′ 12′ 31′ 93′ 124′).

Looking at the quotients recorded on the *ostrakon*, one notices right away that none contains more than six fractions and that no fraction has a denominator above 186. The latter is no trivial achievement, since, for example, calculations involved in producing quotients of 4÷31 as 8′ 248′ or of 13÷31 as 3′ 12′ 372′ would have been easier or resulted in shorter decompositions than those recorded on the *ostrakon*, had no condition been imposed on the value of denominators (see notes to lines 4-5 and 14). It is thus likely that its value was limited, for example, to 200.[20] That such artificial complications could be devised for training purposes in problems with fractions is now confirmed by partition problems in the recently published P.Math. (TM 92734; second half of the 4th cent. A.D., Oxyrhynchus?).[21] For example, Problem C4 there asks for 1 4′ 7′ to be partitioned into seven fractions with a stipulation μὴ πρόβα(ινε) ρ̅, «do not surpass 100», meaning that no denominator in the quotient can be greater than 100.[22] The same condition is implicit in the problems with fractions in P.Cair. cat. 10758 and in calculations recorded in P.Yale IV 187, although it is not spelled out in these texts. Since a division table is essentially a set of computational problems with fractions, it would be not surprising to have similar conditions imposed to make them more challenging, perhaps especially to eliminate «lazy options» afforded by some algorithms.[23]

[20] While it is possible to set a limit of 100 on the value of denominators in division tables with divisors below 19, in division tables for prime divisors from 19 to 31 it is possible to set it at 200.
[21] Bagnall / Jones 2019 is the ed.pr. of this codex, which includes a range of mathematical problems, metrological conversions, and models for contracts.
[22] Bagnall / Jones 2019, 72-73, with conjectural reconstruction of its computation on p. 52.
[23] Bagnall / Jones 2019, 51.

Although only a few lines remain of the table for 57′ on O.Petr.Mus. 65, which, as Azzarello convincingly argues, was inscribed by the same hand as O.Petr.Mus. 64,[24] it is likely that the computation of that table had similar stipulations. All that can be discerned or reconstructed on O.Petr.Mus. 65 are entries for the dividends 6000, followed by those for 1 through 5, and then for 29. The division 29÷57 was the last entry in the table, after which the computer gave up. Since so little of the *ostrakon* survives, it is difficult to draw any conclusion from the remaining quotients about the computational process, except perhaps to note that 2÷57, recorded as 38′ 114′ (line 5), was likely computed following the pattern that $\frac{2}{3} = \frac{1}{2} + \frac{1}{6}$, i.e., that any $2/n$ where n is divisible by 3 ($n = 3m$) can be decomposed as $\frac{1}{2m} + \frac{1}{6m}$. It should be noted that it is possible to produce a full division table for 57 with a stipulation that no decomposition has more than six fractions and that no denominator exceeds 200.[25] Since 57 is a composite number, computing a full division table for it is an easier and more repetitive task than doing so for 31′. Perhaps this is why the student gave up—or may even had been allowed to give up—after getting halfway through it.

Analysis of the quotients on O.Petr.Mus. 64 allowed us to assess how some of the entries were obtained, that is, whether the divisions were computed or produced by the shortcut method of adding one thirty-first to the quotient of the preceding entry. Our computer seems to have been aware of the potential benefits of computing each entry separately, which tends to produce better decompositions (see notes to lines 11-13 and 19-22). Yet, he used the shortcut to obtain as many as ten results, namely for the dividends 4, 6, 9, 12, 15, 17, 21, 25, 27, and 30, where he simply added a 31′ to the quotient of the preceding entry. It is because of this method that the calculation mistake committed in one entry (the division of 5 in line 6) was extended to the quotient in the next (the division of 6 in line 7); and twice it led to a more awkward quotient than a new computation would have likely produced (divisions of 15 and 25 in lines 16 and 26).

The divisions were not revised: Not only the two mistaken results were not corrected (in divisions of 5 and 6), but also, and more significantly, no improvements were made to an entry if a computation of the following quotient made a better option blatantly obvious. One entry illustrates this particularly well: The quotient of 21÷31 (line 22) is given as 2′ 12′ 20′ 31′ 155′ 186′. This was arrived not by computation, but by adding a 31′ to the quotient of 20÷31 (2′ 12′ 20′ 155′ 186′). Our computer then had to compute the division 22÷31 (since the previous quotient contained 31′), obtaining 3″ 31′ 93′. At that moment it should have become clear that the quotient of 21÷31 can be expressed as 3″ 93′ (i.e. the quotient of 22÷31 minus 31′), a decomposition likely preferable to the one he recorded. A similar situation occurred in the division of 6, which could have been easily improved (and thereby

[24] For the ed.pr. and a paleographical discussion, see Azzarello 2008, 167-170.
[25] Since 57 is 19 times 3, it is possible to produce a full division table for it with the same limit on the value of denominators a table for 19 would have had (which actually can be set at 114 as the largest value).

corrected, see notes to lines 6-8) after the quotient for 7 was computed. It is hardly conceivable that, had the table circulated and been copied, such straightforward improvements would not have been implemented. But for the task set for our computer—to compute a full division table for 31 with the condition that no denominator is greater than 200—the results he produced sufficed.

Similarly to solving problems with fractions, such as partitioning 75÷323 into eight unit fractions or one-twelfth into six,[26] computing division tables for 31 or 57, would «have no immediate practical application».[27] The task must have been meant solely for training purposes, as a computational workout for the mind, exercising which would surely improve one's ability to perform all arithmetic operations.

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[26] P.Cair. cat. 10758, Problems 20 and 50, Baillet 1892, 75 and 88-89, correspondingly.

[27] Bagnall / Jones 2019, 51.

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